Lower Bounds on Stabbing Lines in 3-Space[1]

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Abstract

A stabbing line for a set of convex polyhedra is extremal if it passes through four edges and is tangent to the polyhedra containing those edges. In this paper we present three constructions of convex polyhedra with many extremal lines. The first construction shows $\Omega(n^2)$ extremal stabbing lines constrained to meet two skew lines. The second shows $\Omega(n^4)$ extremal lines which are tangent to two polyhedra. The third shows $\Omega(n^3)$ extremal stabbing lines. This last lower bound almost matches the best known upper bound. These lower bounds are relevant for the design and analysis of algorithms constructing the space of stabbing lines.

1 Introduction

The first algorithm for finding line stabbers of a set $\mathcal{B}$ of polyhedra with total complexity n, due to Avis and Wenger [AW87, AW88], has an $O(n^4 \log n)$ time bound. McKenna and O'Rourke [MO88] improve the time complexity to $O(n^4\alpha(n))$, where $\alpha(n)$ is the pseudo-inverse Ackerman function.

Jaromczyk and Kowaluk [JK88] claim an $O(n^3 2^{\alpha(n)})$ upper bound to the complexity of the set of stabbing lines, and an $O(n^3 2^{\alpha(n)} \log n)$ time algorithm to compute such set. The lower bounds presented here show that the analysis in [JK88] holds only in special cases and that some simplification of the basic algorithm proposed in the same paper produces a sharp increase of the running time, in the worst case.

Currently, the best upper bound on the number of extremal stabbing lines for a set of convex polyhedra in 3-space with total complexity n is $O(n^3 2^{c\sqrt{\log n}})$ for a constant c [PS92]. This upper bound is above the cubic lower bound of Theorem 3 by a sub-polynomial factor.

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The algorithms in [MO88, JK88, PS92] find a stabbing line for a set of convex polyhedra by restricting the search on the class of *extremal stabbing lines*. A stabbing line for a set of convex polyhedra is extremal if it passes through four edges and is tangent to the polyhedra containing those edges. Under some general position assumptions the number of extremal stabbing lines is finite. Moreover, the worst case time complexity bound for these algorithms is strictly related to the maximum number of extremal stabbing lines. It is thus of interest for the analysis of these algorithms to have lower bounds on the maximum number of extremal stabbing lines. No previous result on this issue appeared in literature at the best of my knowledge. In this paper we present three constructions of convex polyhedra with many extremal lines.

The first construction (Section 3) shows $\Omega(n^2)$ extremal stabbing lines constrained to meet two skew lines. The second construction (Section 4) shows $\Omega(n^4)$ extremal stabbing lines for which we relax the extremality condition by requiring tangency with two polyhedra. The third construction (Section 5) shows $\Omega(n^3)$ extremal stabbing lines.

The 3-dimensional constructions presented in this paper use some planar constructions similar to those in [Lew80] and [HDK64], where such constructions are used to disprove Helly-type conjectures on families of convex planar sets. Section 2 introduces the basic definitions and recalls some useful properties of orthogonal projections.

2 Basic definitions

Definition 1 *A line l on the plane is an* extremal planar stabber *for a set of segments if it is a stabbing line, touching two vertices in two different segments*

Definition 2 *A line l is a* extremal stabber *for a set of polyhedra if it is a stabbing line, touching four edges and tangent to the polyhedra containing the edges.*

Definition 3 *A line l is a* 2-extremal stabber *for a set of polyhedra if it is a stabbing line, touching four edges and tangent to two of the polyhedra containing the edges.*

The definition of a 2-extremal stabber is a relaxation of the definition of extremal stabber.

A (non horizontal) line l in 3-space is given by the equations:

$$x = az + b \tag{1}$$
$$y = cz + d \tag{2}$$

Equation 1 represents a line on the zx-plane. Equation 2 represents a line on the zy-plane. Fix two orthogonal planes. Any pair of lines, obtained by drawing one line from each plane, determines uniquely a line in 3-space. The operation of forming a line in 3-space from two planar lines is called in this paper *pairing*. Given a line in 3-space we can find its two planar components on the zx-plane and on the zy-plane by orthogonal *projection*. Projections are continuous mappings and preserve connectivity properties. Two lines in 3-space are the same line if and only if the corresponding projections are the same. If a line l on the zx-plane meets a segment s, then any line in 3-space having l as its zx-orthogonal-projection meets the infinite strip obtained by extending s in the direction of the y-axis.

3 Lower bound for constrained stabbers

We consider the case of extremal stabbing lines, which, furthermore, are constrained to meet two given skew lines L_1, L_2. We prove the following lower bound result:

Theorem 1 *There exists a set $\mathcal{B}$ of polyhedra in 3-space of total complexity n, with $\Omega(n^2)$ extremal stabbing lines constrained to intersect two given skew lines.*

Proof. Given a segment s on the zx-plane that does not belong to a line through the origin $A = (0, 0)$, the lines through s and the origin A form a double wedge. Conversely, given a double wedge whose apex is A there exists a segment (not unique) generating that wedge.

We construct a family of $\Theta(n)$ wedges W_{zx} whose intersection has $\Theta(n)$ components. We draw a set of n lines $L = \{l_i\}$ through the origin with slopes $i\pi/2n$ for $i = 0 \ldots n$. Then, we form pairs of lines (l_{2j}, l_{2j+1}) for

$j = 0 \dots \lfloor n/2 \rfloor$. For each pair of lines we obtain four *open* wedges and two double wedges by pairing opposite wedges. We select for each pair of lines the double wedge whose wedges span an obtuse angle. The set of double wedges so obtained, W_{zx}, has $\lfloor n/2 \rfloor$ wedges and the intersection of these wedges has $\lfloor n/2 \rfloor$ connected components. From the set of wedges W_{zx} we can derive easily a set of segments G_{zx} generating W_{zx}. We have the following property: the set of planar stabbing lines for $G_{zx} \cup \{A\}$ has $\lfloor n/2 \rfloor$ connected components, each one having two extremal planar stabbing lines at its boundary. We generate a similar set of segments G_{zy} on the zy-plane, this time choosing as constraining point $A' = (0, 1)$.

We extend orthogonally in 3-space the segments in G_{zx} along the y-axis direction and G_{zy} along the x-axis direction. We obtain two sets of *infinite slabs*. Similarly we extend the point A in the direction of the y-axis and the point A' in the direction of the x-axis. We obtain two skew lines L_1 and L_2. By choosing on the zx-plane n the extremal planar stabbing lines for $G_{zx} \cup \{A\}$, and pairing them with n extremal planar stabbing lines for $G_{zy} \cup \{A'\}$ on the zy-plane we obtain $\Omega(n^2)$ extremal stabbing lines for the set of slabs, constrained to meet two skew lines L_1 and L_2. ■

An $O(n^2)$ upper bound for the number of extremal stabbers constrained through two lines is easily derived from a discussion in [CEGS89].

4 Lower bound for 2-extremal stabbing lines

It is useful to introduce the following notation: for a set of segments S and a point p, $\mathcal{W}(p, S)$ is the set of double wedges formed by the set of lines through p and each segment in S. With this notation, the set W_{zx} in the proof of Theorem 1 is equal to $\mathcal{W}(A, G_{zx})$.

Theorem 2 *There exists a set $\mathcal{B}$ of convex polyhedra in R^3 of total combinatorial complexity n with $\Omega(n^4)$ 2-extremal-stabbing lines.*

Proof. Consider the set of segments G_{zx} generated in the proof of Theorem 1, and the planar arrangement $\mathcal{A}(G_{zx})$ formed by connecting *every pair of end-points of segments* in G_{zx} with a line. By choosing G_{zx} with some care we can make sure that the origin A is in the interior of a cell of $\mathcal{A}(G_{zx})$. If we select any other point p in the cell containing A and we construct the set $\mathcal{W}(p, G_{zx})$, this set has the same combinatorial structure of $\mathcal{W}(A, G_{zx})$, in particular the intersection of this new set of wedges has $\Omega(n)$ components. Now we place a convex n-gon P_{zx} with vertices $\{p_1 \dots p_n\}$ in the cell containing A.

We build a set L_{zx} of n^2 lines on the zx-plane. The lines in L^i_{zx} are are planar extremal stabbing lines for the set $G_{zx} \cup \{p_i\}$. We define $L_{zx} = \bigcup_{i=1}^{n} L^i_{zx}$. There are n^2 different lines in L_{zx}. We repeat the same construction on the zy-plane, choosing the point $A' = (0,1)$ as the distinguished point. We obtain a set of n^2 lines L_{zy} on the zy-plane. We extend orthogonally the segments G_{zx} along the direction of the y-axis and G_{zy} along the direction of the x axis into *infinite slabs*. Similarly the two n-gons P_{zx} and P_{zy} are extended into *infinite prisms*. The slabs and the two prisms form the set $\mathcal{B}$ of polyhedra.

We construct a set L of n^4 lines in R^3 by choosing all pairs of lines from L_{zx} and L_{zy}: $L = L_{zx} \times L_{zy}$. The lines in L are pairwise different 2-extremal stabbing lines for the set $\mathcal{B}$. ∎

An $O(n^4)$ upper bound for the number of 2-extremal-stabbers is easily derived from a discussion in [CEGS89]).

Note that in the construction of Theorem 2 we have a linear number of objets of constant complexity and two objects with linear complexity. For the case when we allow only triangles, we get an $\Omega(n^3)$ lower bound to the number of 2-extremal-stabbers. This is not surprising because, for triangles in general position, a line touching an edge of a triangle is also tangent to it, therefore a 2-extremal-stabbing line is also a an extremal stabbing line.

5 Lower bound for extremal stabbing lines

Theorem 3 *There exists a set $\mathcal{B}$ of polyhedra in R^3 of total complexity n, with $\Omega(n^3)$ extremal stabbing lines.*

Proof. We consider the set of segments G_{zx} generated in the proof of Theorem 1 and the cell of the arrangement $\mathcal{A}(G_{zx})$ containing A. We place a segment u_{zx} completely contained in such cell, on the x-axis, through A. The space of planar stabbing lines for $G_{zx} \cup \{u_{zx}\}$ has n components and, moreover, for any point on u_{zx} there is a stabbing line from each component through that point. Similarly on the zy-plane we build a set G_{zy} and a segment u_{zy} on the y-axis. We lift the sets G_{zx} and G_{zy} into slabs. From the two segments u_{zx} and u_{zy} we generate a rectangle R_{xy} in 3-space such that u_{zx} is the y-projection of R_{xy} and u_{zy} is its x-projection. We inscribe an n-gon P within R_{xy}. For each vertex p^i of P, let p^i_{zx} (resp. p^i_{zy}) be the orthogonal projection on the zx-plane (resp. zy-plane)

Using the same argument as for Theorem 1 we take n different extremal planar stabbing lines for $G_{xy} \cup \{p^i_{xy}\}$ on the zx-plane. Similarly we obtain on the zy-plane n planar extremal stabbing line for $G_{zy} \cup \{p^i_{zy}\}$. By pairing we obtain n^2 extremal stabbing lines in 3-space for the set of slabs augmented with P. Repeating this construction for each vertex of P we obtain $\Omega(n^3)$ extremal stabbing lines. ∎

The paper [CEGS89] describes a lower bound to the complexity of the *upper envelope of a set of lines* which can be transformed easily into a construction of triangles, giving an $\Omega(n^3)$ bound on the number of extremal stabbing lines. The proof in [CEGS89] is quite complex when all the details of the construction are considered.

6 Notes and open problems

- In [JK88] Jaromczyk and Kowaluk introduce a general technique called skew projection, which is applied in [JK88] to the problem of building (a representation of) the set of lines stabbing a set of polyhedra. As follows from the lower bound of Theorem 1 the analysis of the algorithm in [JK88] holds only for special cases. In the worst case the analysis gives a running time $O(n^4\alpha(n))$.

- In view of Theorem 2, a variation of the basic algorithm in [JK88], which amounts to finding all 2-extremal-stabbing lines, could use $\Omega(n^4)$ time on some carefully constructed problem instance.

- The constructions of Theorems 1 and 2 can be modified so that they exhibit a set of mutually disjoint polyhedra. It is still open the question whether a construction of disjoint polyhedra can attain a cubic number of extremal stabbing lines.

- The construction of Theorem 1 shows a set of polyhedra such that the set of stabbing lines has $\Omega(n^2)$ connected components whenever a natural parametrization of the line is used. It is still open the problem of a matching upper bound on the number of components for the set of stabbing lines.

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