Measuring lines in space - A collection of results

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Abstract

Integral geometry of lines in space has been a key tool in deriving new algorithms for solving integral equations in diverse fields of applied mathematics and computer science such as numerical electrostatics and computer graphics. Here we review those results highlighting the role of integral-geometric transformations.

1 Introduction

The *discrete* and the *continuous* point of views in mathematics are both sources of beautiful results, and often it is convenient to alternate between them to gain insight on the problems at hand.

Combinatorial properties of *finite* sets of lines in 3-space have been a focus of attention in computational geometry since the early 90's (see (Edelsbrunner, 1991) and (Pellegrini, 1997b)) and combinatorics of hyperplane arrangements in the appropriate space (notably 5-dimensional Plücker space) play a dominant role. Conversely, the study of measure-related properties of *infinite* sets of lines require tools and concepts from Integral Geometry. Lines in 3-space are not the most natural primitive object one would think of: points and planes in 3D are rather more natural objects. However, in several applications ranging from Computer Graphics to Electrostatics and Tomography there is much to be gained when lines and their integral geometric properties are placed in the forefront. To give substance to this point of view we will survey some recent results obtained by reasoning according to those schemes. More specifically we will refer to the papers (Pellegrini, 1997a), (Pellegrini, 1998), (Finocchiaro, Pellegrini, and Bientinesi, 1998), (Pellegrini, 1999a) and (Pellegrini, 1999b).

The final applications come from the rather diverse areas of numerical electrostatics and computer graphics, however a common general pattern emerges. There is an initial phase in which a well known mathematical formulation of a problem (an integral equation to be solved) is recast in terms of integrals over sets of lines. In the second phase a Galerkin-type of discretization is applied, resulting in the formulation of a linear system of algebraic equalities. The entries of the corresponding matrix are high dimensional multiple integrals. At this point exploiting several expressions of the differential measure of lines in space we separate the integral expression into an integral on the unit sphere

(2-dimensional) and a kernel that is computed exactly. We apply a Gaussian adaptive strategy for numerical integration on the sphere which gives probably close approximations to the entries of the matrix.

2 Electrostatics

2.1 An Integral Geometric Theory of Electrostatic Fields

In this section we present a geometric interpretation of electrostatic force fields using tools from integral geometry. We adopt an axiomatic approach. We give an integral geometric definition of a field $\vec{G}$, which we will call the *geometric field*. We will show that the geometric field satisfies Gauss's Law of flux through a closed (convex) surface. Well known arguments of electrostatics can then be invoked to base the classical theory of electrostatic fields on Gauss law (Jackson, 1975) (Landau and Lifshitz, 1980). The theory is developed for fully dimensional convex bodies in 3-space.

Let us denote with L an oriented line in 3 space, with $\vec{L}$ the unit vector along L and dL the differential measure of lines in 3-space. Let us consider a density function $\rho_1(p)$ defined over the points p of a compact convex body B_1, and let $\bar{\rho}_1$ be the maximum of the absolute value of $\rho_1(p)$ in B_1. Let us define for a line L the quantity $m_1(L)$:

$$m_1(L) = \int_{p \in L} \rho_1(p) dp.$$

Notice that $|m_1(L)| \leq \bar{\rho}_1 \mu(L \cap B_1)$, where μ measures the length of a segment, and equality holds for a constant density of charge. Let Q_1 be the total charge of B_1:

$$Q_1 = \int_{p \in B_1} \rho_1(p) dp.$$

The Geometric field at a point $p \notin B_1$ is defined as

$$\vec{G}_1(p) = \int_{L \cap p \neq \emptyset} m_1(L) \vec{L} dL. \tag{1}$$

The lines are considered oriented from B_1 to p, and the integral is an integral of vectors. Since we consider convex bodies, although the lines are oriented, we use in many cases the underlying set of unoriented lines when it is convenient to do so. Sometimes it is more convenient to work with the component of $\vec{G}$ is some direction $\vec{w}$, which is given by

$$\vec{G}_1 \cdot \vec{w} = \int_{L \cap p \neq \emptyset} m_i(L)(\vec{L} \cdot \vec{w}) dL.$$

We will compare our results with those obtained through the traditional definition of the electrostatic field in the Gauss unit system (see (Jackson, 1975)).

2.1.1 *Gauss law*

Gauss law states that the integral of the component of the electric field $\vec{E}(p)$ in direction $\vec{n}(p)$ normal to a closed connected surface $S = \partial B$ is equal to 4π times the total charge in the bounded connected region B enclosed by the surface S:

$$\int_{p\in S} \vec{E}\cdot\vec{n}dp = 4\pi\int_{p\in B}\rho(p)dp.$$

Let us take a convex surface S and let us parameterize the (directed) lines as a point on the surface and an (outer) direction u in the set of directions $U/2$. In these coordinates the differential element of lines is $dL = \cos\theta dpdu$, where dp is the differential element of surface area, du is the differential element of directions, and θ in the angle between the (outer) normal to $\vec{n}(p)$ and the direction u. We have that

$$\int_{L\cap S\neq\emptyset} m_1(L)dL = \int_{p\in S}\int_{u\in U/2} m_i(L)\cos\theta dudp = \int_{p\in S}\vec{G}(p)\cdot\vec{n}(p)dp,$$

where the last expression is the standard definition of the flux of a vector field through the surface S. Now we start from the same integral but we identify any line by a pair $L = (u, q)$, where u is the direction of L, and q is the point intercepted by L on a plane $P(u)$ of normal u and incident to the origin. Using these coordinates the differential measure of lines is $dL = du\wedge dq$ (Santaló, 1976). Consider $B_1\subseteq B$. We obtain

$$\int_{L\cap S\neq\emptyset} m_1(L)dL = \int_{L\cap B_1\neq\emptyset} m_1(L)dL = \int_{u\in U}[\int_q m_1(u,q)dq]du =$$
$$\int_{u\in U}[\int_q[\int_{p\in L, L=L(u,q)}\rho_1(p)dp]dq]du = \int_{u\in U}[\int_{p\in B_1}\rho_1(p)dp]du = 4\pi Q_1,$$

where in the last integral the domain of q is the projection of B onto $P(u)$. To summarize, starting from the same integral we have derived two different expressions which we may therefore equate, obtaining

$$\int_{p\in S}\vec{G_1}\cdot\vec{n}dp = 4\pi Q_1,$$

which is the statement of Gauss's law for the field $\vec{G}$.

2.1.2 *Force acting between two convex bodies*

From now on we will not distinguish any more between the geometric field $\vec{G_i}$ and the electrostatic field $\vec{E_i}$ generated by a convex body B_i. Let us denote with $\vec{F_{12}}$ the force acting between bodies B_1 and B_2. We can express this force by integrating over the *points* of B_1 the value of the electrostatic field generated by B_2 at each such point. Formally we have

$$\vec{F_{12}} = \int_{p\in B_1}\vec{E_2}(p)\rho_1(p)dp,$$

which we expand using (1) into

$$\vec{F}_{12} = \int_{P \in B_1} \left[\int_{L \cap p \neq \emptyset} m_2(L) \vec{L} dL \right] \rho_1(p) dp.$$

Now we transpose the two integrals obtaining

$$\vec{F}_{12} = \int_L \left[\int_{L \cap p \neq \emptyset, p \in B_1} m_2(L) \rho_1(p) dp \right] \vec{L} dL.$$

The function $m_2(L)$ in the inner integral does not depend on p, thus we can take it out of the inner integral. The inner integral that is left is $m_1(L)$.

The total force acting between B_1 and B_2 is thus reduced to a single integral over the set of lines in 3-space:

$$\vec{F}_{12} = \int_L m_1(L) m_2(L) \vec{L} dL. \tag{2}$$

Now we identify any line by a pair $L = (u, q)$, where u is the direction of L, and q is the point intercepted by L on a plane $P(u)$ of normal u and incident to the origin. Using these coordinates the differential measure of lines is $dL = du \wedge dq$ (Santaló, 1976). We obtain the new formula

$$\vec{F}_{12} = \int_L m_1(L(u,q)) m_2(L(u,q)) \vec{u} dq du.$$

Thus we can split the computation into an integration over the set $U/2$ of unoriented directions[1], and, for a fixed direction $u \in U/2$, an integration over a planar set of points. So, finally

$$\vec{F}_{12} = \int_{u \in U/2} V_{12}(u) \vec{u} du, \tag{3}$$

where the kernel function V_{12} is

$$V_{12}(u) = \int_q m_1(L(u,q)) m_2(L(u,q)) dq. \tag{4}$$

Remark. The function $V_{12}(u)$ does not diverge. This is a main advantage with respect to more traditional formulations of the force acting between two bodies that are based on integrating a kernel function, such as $1/r^2$ or $1/r$, which diverge when r goes to zero.

2.2 Robin's equation

Robin's integral equation (5) describes the equilibrium conditions for the surface charge density $\sigma(p)$ on the surface $S = \partial D$ of a closed compact (but not necessarily connected) 3-dimensional domain D (Cade, 1995). In this section we show a possible discretization of Robin's integral equation resulting in a Boundary Element formulation. We are interested in showing the typical form of the entries of the stiffness matrix. Then in Subsection 2.2.1

[1] The set of directions U is represented by the unit sphere in 3-space. We obtain the set of unoriented directions by identifying antipodal points on the unit sphere.

we turn such entries into geometric integrals without singularities. In the last section we point at an algorithm for approximating such geometric integrals with probable error bounds. Robin's equation for the surface charge density function $\sigma(p)$ over a surface S is

$$2\pi\sigma(p) = \int_{p' \in S} \frac{\cos\theta(n(p), pp')}{r^2}\sigma(p')dp', \tag{5}$$

where $\theta(n(p), pp')$ is the angle formed by the normal $n(p)$ to S at p with the line p, p', and r is the distance between p and p'. We now consider the surface as partitioned in polygons P_i with index $i \in [1, \ldots, k]$, and we compute the force acting on each polygonal face P_i. Since the electric field is normal at any point of the polygon, and the normal direction to each polygon is constant, the resultant force is also normal to the polygon P_i. Let F_i be the modulus of the (normal) force acting on polygon P_i. We can compute F_i by using the well known relation $E(p) = 2\pi\sigma(p)$ between the module of the field and the charge density at any point of the surface S. With σ_i we denote the restriction of the function σ on P_i. We obtain

$$F_i = \int_{p \in S} E_i(p)\sigma_i(p)dp = \int_{p \in S} 2\pi\sigma_i(p)\sigma_i(p)dp \tag{6}$$

If we use the right side side of the equation (5) we obtain

$$F_i = \int_{p \in S} E_i(p)\sigma_i(p)dp = \int_{p \in S} \left[\sum_{j=1, i \neq j}^{k} \int_{p' \in S_j} \frac{\cos\theta(n(p), pp')}{r^2}\sigma_j(p')dp' \right] \sigma_i(p)dp \tag{7}$$

Now we expand each local density function σ_i in an orthonormal basis with unknown scalar coefficients $A_{i,h}$:

$$\sigma_i(p) = A_{i,0}\sigma_0(p) + A_{i,1}\sigma_1(p) + A_{i,2}\sigma_2(p) + \ldots \quad .$$

Denoting with $H_{i,h}$ the following integral

$$H_{i,h} = 2\pi \int_{p \in P_i} \sigma_h(p)\sigma_h(p)dp,$$

and exploiting orthonormality[2], we obtain that formula (6) for F_i becomes

$$F_i = H_{i,0}[A_{i,0}]^2 + H_{i,1}[A_{i,1}]^2 + H_{i,2}[A_{i,2}]^2 + H_{i,3}[A_{i,3}]^2 + \ldots \quad .$$

Expanding formula (7) for F_i we obtain

$$F_i = \sum_j \sum_h \sum_k \int_{p \in S_i, p' \in S_j} \frac{\cos\theta(n(p), pp')}{r^2}\sigma_h(p)\sigma_k(p')A_{i,h}A_{j,k}dpdp'.$$

The above identities are satisfied by the solution of this system:

$$H_{i,h}A_{i,h} = \sum_j \sum_k A_{j,k} \int_{p \in S_i, p' \in S_j} \frac{\cos\theta(n(p), pp')}{r^2}\sigma_h(p)\sigma_k(p')dpdp'$$

[2]The integral of product of functions with different index is null.

The unknowns in the resulting linear system are the values of $A_{i,h}$. The number of such variables, sometimes referred to as the "dimension" of the system, is the product of the number of polygons P_i and the number of terms of the expansion of the density functions, in case of a uniform expansion on all the patches (also known as h-Galerkin method). Alternatively we might truncate the basis expansions differently on each polygon P_i so the total number of unknown is the sum of the terms in the truncated functional expansions (also known as hp-Galerkin). The coefficients $H_{i,h}$ can be pre-computed easily with exact analytic integration. The other coefficients on the right sides of the equations are of the form

$$C_{i,h,j,k} = \int_{p \in S_i, p' \in S_j} \frac{\cos\theta(n(p), pp')}{r^2} \sigma_h(p)\sigma_k(p')dpdp', \tag{8}$$

where the functions σ_h and σ_k are known elements of the functional basis. Next we show how we can compute such coefficients using the general techniques developed in the first part of the paper. Integrals of this form (8) fall within the class of integrals studied in (Sauter and Schwab, 1997). They are obtained in that paper with a double-layer potential formulation of the equilibrium of conductors.

2.2.1 *Integral geometric transformation*

In this section we apply integral geometric transformations to the integral in (8). To simplify the notation we drop subscripts inherited from the discretization process and we consider two triangles T_1 and T_2 and known densities σ_1 and σ_2. Thus the integral we are considering is

$$C_{12} = \int_{p \in T_1, p' \in T_2} \frac{\cos\theta(n(p), pp')}{r^2} \sigma_1(p)\sigma_2(p')dpdp'. \tag{9}$$

Let us call $\phi(n(p'), pp')$ the angle formed by the line pp' with the normal to T_2 at p'. Let us assume for the moment being that the cosine of such angle is not null. Multiplying and dividing in (8) by the cosine of this angle we obtain

$$\int_{p \in T_1, p' \in T_2} \frac{1}{\cos\phi(n(p'), pp')} \frac{\cos\theta(n(p), pp')\cos\phi(n(p'), pp')}{r^2} \sigma_1(p)\sigma_2(p')dpdp'.$$

Now we can turn to a geometric integral since we have isolated an expression of dL:

$$\int_{L, L\cap T_1 \neq \emptyset, L \cap T_2 \neq \emptyset} \frac{1}{\cos\phi(n(p'), p, p')} \sigma_1(p)\sigma_2(p')dL. \tag{10}$$

Next we express a line L in the (u, q) coordinates:

$$\int_{u \in U} \left[\int_q \frac{1}{\cos\phi(q, u)} \sigma_1(p)\sigma_2(p')dq \right] du.$$

For flat polygonal faces and a fixed direction u, the value $\cos\phi(q, u)$ does not depend on q and can be taken out of the inner integral sign:

$$C_{12} = \int_{u \in U} \left[\frac{1}{\cos\phi(u)} \int_q \sigma_1(p)\sigma_2(p')dq \right] du. \tag{11}$$

Let us call $K'(u)$ the kernel integral

$$K'(u) = \frac{1}{\cos\phi(u)} \int_q \sigma_1(p)\sigma_2(p')dq. \tag{12}$$

In (Pellegrini, 1998) it is shown that the value of $K'(u)$ does not diverge when $\cos\phi(q,u)$ tends to zero, and the value of the limit function can be computed with a formula obtained by a limiting process applied to formula (3).

Remark. Alternatively, we can also obtain formula (10) by considering the formula (3) in the first part of the paper applied to two prisms with basis T_1 and T_2, and by letting the height of the prisms go to zero while maintaining a consistency condition on the local charge density.

3 Computer graphics: Rendering equation revisited

The rendering equation describes the exchanges of radiant energy in a given 3D scene. One of the main obstacle to an efficient numerical solution is attributed to the *non-local* effects due to *occlusions*.

3.1 Standard continuous forms of the rendering equation

We denote with Σ be the collection of surfaces in the scene and with Ω the unit sphere of direction. Let L be the radiance function, E the emitted radiance function (which is non-zero for light sources), and ρ the *Bi-directional reflectance density function BRDF* modelling the local behaviour of light on the surfaces. Two standard forms of the rendering equations are reported in (Sillion and Puech, 1994; Cohen and Wallace, 1993). The first form is

$$L(y,\omega) = E(y,\omega) + \int_{\eta\cdot n_y>0} \rho(y,\omega,\eta)L(x(y,\eta,\Sigma),-\eta)(\eta\cdot n_y)d\eta, \tag{13}$$

where y and x are points on the surface Σ; ω and η are directions (points on the unit sphere Ω); $x(y,\omega,\Sigma)$ is the first point of Σ hit by a ray from y in direction ω; and n_y is outer unit vector at y normal to Σ. The second form is

$$L(y,z) = E(y,z) + \int_{x\in\Sigma} \rho(x,y,z)L(x,y)G(x,y)dx, \tag{14}$$

where x, y, z are points on Σ, and

$$G(x,y) = \frac{\cos\theta_x \cos\theta_y}{|x-y|^2} V(x,y,\Sigma), \tag{15}$$

where θ_x (resp. θ_y) is the angle formed by the outer normal n_x (resp. n_y) and the line joining x and y, while the function $V(x,y,\Sigma)$ has value 1 if the open segment xy does not intersect Σ, and zero otherwise.

3.2 Simulating occlusions by superimposition

In this section we show how to modify the BRDF function describing the local behaviour of light on the surface in order simulate occlusions by superimposition. The main idea is that energy arriving onto a surface from a direction η produces a negative energy flux in direction $-\eta$ so that behind the surface their sum is null. Naturally this annihilation persists when multiple surfaces are considered on the same ray.

From a mathematical point of view this modified BRDF makes radiosity calculations much more similar to BEM for conductors in electrostatics where the superimposition principle is used. We will show that the occlusion effect is achieved on any single line, thus the cancellation effect carries over on integrals of lines.

We start from a standard form of the rendering equation (13). Let us call U_+ the halfsphere, where $\eta \cdot n_y > 0$. We integrate contribution of incoming light from U_+ since the assumption is that light outside the objects cannot go inside and vice versa. So, to describe the balance of light outside the objects, we assume no sources inside them. So ρ is defined and used for $\eta \in U_+$, but also ω is considered and used in U_+ because by default it is assumed that all the light received from outside is re-sent outside. So, $\omega \in U_+$. We are thus free to decide the value to associate with directions in the complementary domain U_-, and we use this freedom to simulate occlusions by means of a special BRDF defined on U_-. So, we define, for $\omega \in U_-$ and $\eta \in U_+$,

$$\rho(y, \omega, \eta) = -\frac{\delta(\omega - (-\eta))}{|n_y \cdot \eta|},$$

where δ is the delta Dirac function. Thus carrying out easy calculation from (13) we obtain $L(y, -\eta) = -L(x, -\eta)$, which shows the negative flux. Note that L is preserved in absolute value through the surface but changes sign. For $\eta \in U_-$, we have $\rho(y, \omega, \eta) = 0$.

Intuitively for rays coming from inside an object all contributions cancel off anyway. So, we do not need to count them. We have to modify the rendering equation and collect contributions from all the intersection points along a ray, not just the visible one. The new form of the rendering equation is

$$L(y, \omega) = E(y, \omega) + \int_{\eta \cdot n_y > 0} \rho'(y, \omega, \eta) \left[\sum_{m=1}^{v} L(x_m(y, \eta), -\eta) \right] |\eta \cdot n_y| d\eta, \qquad (16)$$

where $x_1 \dots x_v$, is the collection of points of intersection between the ray from y in direction η on the surfaces in Σ. With ρ' we denote the BRDF modified as above.

An alternative way of viewing the superimposition is to use formulae of inclusion/exclusion along each line instead of describing explicitly the combinatorial structure of the visible surfaces.

3.3 Derivation of the new rendering equation

We start our manipulations from formula (16). Let (y, ω) define the oriented line l, and let (y, η) define the oriented line h, then easily $(x, -\eta)$ defines the oriented line $-h$. This notation will be important later to distinguish between the lines and their representation using a coordinate system. We assume now an expansion of the function L into a functional basis. Specifically we consider a covering of the domain

$$\Sigma \times \Omega = \cup_{i=1}^{n} (\Sigma_i \times \Omega_i).$$

Let $f_{i,j}(\cdot, \cdot)$ be a basis function with compact support in $\Sigma_i \times \Omega_i$ (i.e., it is identically zero in $(\Sigma \times \Omega) \setminus (\Sigma_i \times \Omega_i)$). We express the approximate solution $\tilde{L}$ with $\tilde{L} = \sum_{i,j} a_{i,j} f_{i,j}$ for unknown reals $a_{i,j}$. We now formally multiply equation (16) by the weight function $f_{i,j}$ and we compute the weighted residual which is then set to zero. Practically we multiply by $f_{i,j}$ and integrate on the differential surface element dy around y, for $y \in \Sigma_i$; and by the differential element $d\omega$ of the angle $\omega \in \Omega_i$.

$$f_{i,j}(y,\omega) L(y,\omega) dy d\omega =$$
$$f_{i,j}(y,\omega) E(y,\omega) dy d\omega + \left[\int_{\eta \cdot n(y) \geq 0} \rho'(y,\omega,\eta) [\sum_m L(x_m, -\eta)] |\eta \cdot n(y)| d\eta \right] f_{i,j}(y,\omega) dy d\omega.$$

Next we integrate over the domain $\Sigma \times \Omega$. However due to the choice of support for the functions $f_{i,j}$ this is equivalent to integrate over $\Sigma_i \times \Omega_i$:

$$\int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y,\omega) L(y,\omega) dy d\omega = \int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y,\omega) E(y,\omega) dy d\omega +$$
$$\int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} \int_{\eta \cdot n(y) \geq 0} \rho'(y,\omega,\eta) [\sum_m L(x_m, -\eta)] |\eta \cdot n(y)| d\eta f_i(y,\omega) dy d\omega.$$

Up to now the derivation has been quite standard. Since $|\eta \cdot n(y)| d\eta dy$ is the differential measure of lines around h, dh, for h meeting Σ_i (Santaló, 1976). Thus we can rewrite more compactly

$$\int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y,\omega) L(y,\omega) dy d\omega =$$
$$\int_{y \in \Sigma_i} \int_{\omega \in \Omega_i} f_{i,j}(y,\omega) E(y,\omega) dy d\omega + \int_{\omega \in \Omega_i} \int_{h \cap \Sigma_i \neq \emptyset} f_{i,j}(y,\omega) \rho'(y,\omega,\eta) [\sum_m L(-h_m)] dh d\omega,$$

where by h_m we denote the pair (x_m, η). Since $L(-h_m)$ does not depend on ω we can exchange the order of integration and take $L(-h_m)$ out of the integration in $d\omega$:

$$\int_{h \cap \Sigma_i \neq \emptyset} [\sum_m L(-h_m)] \left[\int_{\omega \in \Omega_i} \rho'(y,\omega,\eta) f_{i,j}(y,\omega) d\omega \right] dh.$$

We assume that we have chosen the functions ρ' and $f_{i,j}$ in a class for which we are able to integrate analytically in $d\omega$. More precisely we assume the following function to be computable analytically:

$$Q_{i,j}(h) = Q_{i,j}(y,\eta) = \int_{\omega \in \Omega_i} f_{i,j}(y,\omega) \rho'(y,\omega,\eta) d\omega. \tag{17}$$

Polynomials naturally satisfy this property.

3.3.1 *General functional basis*

We assume that for each domain $\Sigma_i \times \Omega_i$ we have either one function f_i or several orthonormal functions $f_{i,j}$. In either case we discuss the terms of equation (17):

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)L(y,\omega)dyd\omega = a_{i,j}\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(y,\omega)^2 dyd\omega$$

For short we denote with $H_{i,j}$ the coefficient of $a_{i,j}$:

$$H_{i,j} = \int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_i(y,\omega)^2 dyd\omega,$$

which we suppose to be able to integrate analytically. The emitted energy is weighted by the function $f_{i,j}$:

$$\int_{y\in\Sigma_i}\int_{\omega\in\Omega_i} f_{i,j}(\omega,y)E(y,\omega)dyd\omega = E_{i,j},$$

and we are able to compute $E_{i,j}$ analytically when it is a polynomial. Summarizing,

$$a_{i,j}H_{i,j} = E_{i,j} + \int_{h\cap\Sigma_i\neq\emptyset}[\sum_m L(-h_m)]Q_{i,j}(h)dh.$$

Using the expansion of $L(-h)$ we get

$$a_{i,j}H_{i,j} = E_{i,j} + \int_{h\cap\Sigma_i\neq\emptyset}\left[\sum_{k\neq i,s} a_{k,s}f_{k,s}(-h)\right]Q_{i,j}(h)dh.$$

Rearranging,

$$a_{i,j}H_{i,j} = E_{i,j} + \sum_{k\neq i,p} a_{k,s}\left[\int_{h\cap\Sigma_i\neq\emptyset} f_{k,s}(-h)Q_{i,j}(h)dh\right].$$

Now we restrict the integration domain to lines for which $f_{k,s}$ is not zero. In this mmaner, we obtain the following expression for the coefficient $K_{ij,ks}$ of $a_{k,s}$:

$$K_{ij,ks} = \int_{h\cap\Sigma_i\neq\emptyset} f_{k,s}(-h)Q_{i,j}(h)dh = \int_{h\cap\Sigma_i\neq\emptyset,-h\cap\Sigma_k\neq\emptyset,Dir(-h)\in\Omega_k} f_{k,s}(-h)Q_{i,j}(h)dh,$$

where $Dir(\cdot)$ denotes the direction of a line. Note that here the function $Q_{i,j}$ carries contribution from the BDRF and from the basis function $f_{i,j}$. We express $h = (q,u)$ and we obtain the following expression for $K_{ij,ks}$:

$$K_{ij,ks} = \int_{-u\in\Omega_k}\int_q f_{ks}(-u,q)Q_{ij}(u,q)dqdu. \tag{18}$$

To apply the technique in (Pellegrini, 1999a) we make the assumption that, for any fixed u, $f_{ks}(-u,q)$ is polynomial in local Cartesian coordinates for q. This assumption is easy to satisfy by using a product polynomial basis. Also $Q_{ij}(u,q)$ needs to satisfy the same condition.

Finally the domain of the external integration is restricted to direction such that $-u \in \Omega_k$. If we restrict the possible shapes for Ω_k to convex spherical polygons (therefore a shape bounded by arcs of great circles on Ω) we can accommodate the range reduction within the spherical domains considered in (Pellegrini, 1999a). Somewhat simpler formulae are derived for a piecewise-constant functional basis often used in applications.

4 Numerical evaluation

We give here the relevant concepts for the numerical evaluation of the integrals obtained as a result of the geometric transformations. Proofs and algorithms are in (Pellegrini, 1998) and (Pellegrini, 1999a). The following definition from (Pellegrini, 1998) capture the notion of a *locally analytic* function on the sphere for which we can use powerful Gaussian integration techniques.

Definition 1 *Integral:* $\int_{u \in U} f(u)du$ *is* **well-behaved** *if it can be represented as:*

$$\sum_j \int_{u \in D_j} f_j(u)du,$$

where for each j, $D_j \subset U$, *and the collection of the domains* D_j *is a partition of* U. *Moreover, for each* j, *we can find a local system of reference* LSR_j *of parameters* α_j *and* β_j *such that:*

1. *In* LSR_j,
$$D_j = I_{j,\alpha} \times I_{j,\beta} = [\alpha_{j,0}, \alpha_{j,1}] \times [\beta_{j,0}, \beta_{j,1}]$$
That is, D_j *is mapped to a bounded rectangular domain in* LSR_j.

2. *By substitution* $f_j(u)$ *is mapped to* $f_j(\alpha_j, \beta_j)$ *and we have the invariant differential element of directions* $du = g_j(\alpha_j, \beta_j)d\alpha_j d\beta_j$.

3. *For every* $\bar{\alpha} \in I_{j,\alpha}$, *the restriction* $f_j(\bar{\alpha}, \beta_j)g(\bar{\alpha}, \beta_j) : I_{j,\beta} \to R$ *admits an analytic extension in an open rectangle* $Rect(\bar{\alpha})$ *of the complex plane which contains strictly* I_β.

4. *Fore every* $\bar{\beta} \in I_{j,\beta}$, *the restriction* $f_j(\alpha, \bar{\beta}_j)g_j(\alpha, \bar{\beta}) : I_{j,\alpha} \to R$ *admits an analytic extension in an open rectangle* $Rect(\bar{\beta})$ *of the complex plane which contains strictly* I_α.

By set A strictly containing B we mean that A contains B and the boundary of A is disjoint from the boundary of B. The following result is established in (Pellegrini, 1998) and (Pellegrini, 1999a):

Theorem 1 *Integrals (2), (11) and (18) are well-behaved.*

For the computation we use the following Theorem whose proof (and corresponding algorithm) is in (Pellegrini, 1998):

Theorem 2 *A well-behaved integral can be approximated using $O(n)$ Gaussian points while achieving absolute error $O(c^{-\sqrt{n}})$, for some $c > 1$.*

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