Lesson 5. Voronoi Diagrams and Delaunay Triangulations

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1 Disclaimer

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2 Introduction

Here we introduce four geometric constructions and we will show that we can derive easily all of them from one of them, thus basically one algorithms will suit all. Material mostly taken from [Ede87].

3 Voronoi Diagrams

Let S be a finite set of n points in R^d,a nd let $D : R^d \times R^d \to R$ be the Euclidean distance function.

For any point $p \in R^d$, call $C(p)$ the subset of S that minimizes $D(p,s)$ for $s \in S$.

Definition 1 *The "abstract" Voronoi Diagram is* $Vor_A(S) = \{C(p)|p \in R^d\}$.

Note that $Vor_A(S) \subseteq 2^S$ is a subset of the powerset of S.

Definition 2 *The Voronoi Diagram* $Vor(S)$ *is the cell complex whose faces are equivalence classes of points in the Abstract Voronoi Diagram.*

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That is every cell in $Vor(S)$ is the set of points in R^2 such that $C(p) = C$ for a given $C \in Vor_A(S)$.

It is easy to see that in 2D the cell complex is a straight planar map with $O(|S|)$ cells.

4 Convex Hull

Definition 3 *Given a finite set $S \subset R^d$ the convex hull of S, $CH(S)$ is the set of points $q \in R^d$ such that for positive scalars $\alpha_1, .., \alpha_d$, such that $\sum_{i=1,d} \alpha_i = 1$, we have $q = \sum_{i=1,d} p_i \alpha_i$ for some $p_i \in S$.*

in other words, every point in $CH(S)$ can be expressed as the convex combination of d points in S. There are many equivalent definitions. One is:

Definition 4 *Given a finite set $S \subset R^d$ the convex hull of S, $CH(S)$ is the intersection of all closed halfspaces that contain S.*

In particular the last definition make it obvious that $CH(S)$ is a convex bounded polyhedron in R^d and therefore it is also a polytope.

5 Intersection of halfspaces

Given a set $H = \{h_1, ..h_n\}$ of n closed halfspaces in R^d, compute their intersection $P = \bigcap_{h_i \in H} h_i$. Note that it is possible that $P = \emptyset$.

6 Delaunay Triangulation

Definition 5 *Given a set S of points in the plane, a planar straight line map $T(S)$ is called a triangulation if*

(i) $R^2 \setminus CH(S)$ is the only unbounded cell of the map.

(ii) S is the set of vertices of the map.

(iii) Every bounded cell of the map is a triangle.

Definition 6 *Given a set S of points in the plane, a planar straight line map $DT(S)$ is called the Delaunay Triangulation if for any edge (u, v) in $DT(S)$ there is a circle through u and v not containing any other point of S.*

Delaunay triangulations can be defined differently and have many interesting properties.

For us it is interesting that:

Property 1 *Among all Triangulations $T(S)$ the Delaunay Triangulation $DT(S)$ is maximal in the lexicographic order of the increasing angle sequences.*

This property is important when we care that the triangulation avoids very small angles (for example to in uses in numerical integration).

Property 2 *Given S, $(u, v) \in S \times S$ is an edge of the Delaunay Triangulation if and only if the Voronoi cell of $\{u\}$ is adjacent to the Voronoi Cell of $\{v\}$.*

This implies that the Delaunay Triangulation has an underlying graph that is dual to the Voronoi diagram. Given $Vor(S)$ we can compute in time $O(n)$ $DT(S)$.

7 Polarity

Consider a point $p \in R^d$, we define a mapping from points to halfspaces in R^d as follows.

$$h_p^+ = \{x \in R^d | x \cdot p \leq 1\}$$

also we consider the supporting hyperplane:

$$h_p = \{x \in R^d | x \cdot p = 1\}$$

where $\cdot$ is the inner product. We call h_p^+ the positive polar of the point p. This mapping is well defined for any $p \neq O$ different from the origin.

Note that h_p^+ always contains the origin O properly.

We can visualize it in 2D as the halfplane supported by the line intersecting the segment Op at distance $1/|Op|$ from the origin, and containing the Origin.

The mapping between R^d/O and the set H^+ of halfspaces containing O properly is 1-1. Thus we have mappings: $D : R^d \setminus O \to H^+$ and $D^{-1} : H^+ \to R^d \setminus O$.

Property 3 *Order preservation. If point p belongs to the positive halfspace h then the point $D^{-1}(h)$ belongs to the positive halfspace $D(p)$.*

Property 4 *Incidence preservation. If point p is incident to line l (not through the origin) then line supporting $D(p)$ is incident to the point $D^{-1}(l^+)$.*

Consider a set of halfspaces $I = \{h_1, ..., h_n\}$ such that $P = \bigcap_i h_i$ has interior points. In such case we can translate the origin to an interior point and obtain an equivalent problem in which all halfspaces are positive. Consider the set of points $S = \{D^{-1}(h) | h \in I\}$ and its convex hull $Q = CH(S)$.

Straightforward applications of the incidence and order properties imply that:

Theorem 1 *There is a 1-1 correspondence between k-faces of P and $(d-k-1)$ faces of Q, for $0 \leq k \leq d-1$, that preserves incidences.*

In particular the problem of computing the non-empty intersection of halfspaces is reducible to the problem of computing the convex hull, and vice versa, the problem of computing the convex hull can be transformed into the problem of computing the intersection of positive halfspaces. However we must be careful that the origin is interior to the convex hull by an appropriate translation of coordinates.

8 Lifting to the unit Paraboloid

Consider a set S of n points in R^d with a Cartesian coordinate system with variables vector $x = (x_1, ..x_d)$.

Consider R^d embedded in R^{d+1} as the hyperplane $x_{d+1} = 0$.

Consider the paraboloid $U : x_{d+1} = x \cdot x$ in R^{d+1}.

We map a point $p \in S$ to the hyperplane $h_p : x_{d+1} = 2(p \cdot x) - (p \cdot p)$, which is the hyperplane tangent to the paraboloid at the point of vertical projection of p onto the paraboloid.

Next we show that for any pair of points $p, q \in R^d$ the distance squared between p and q is equal to the distance between the projection points $U(q)$ and $h_p(q)$.

Proof.

Easily

$$d^2(p,q) = (p-q) \cdot (p-q) = (p \cdot p) + (q \cdot q) - 2(p \cdot q)$$

On the other hand:

$$d(U(q), h_p(q)) = (q \cdot q) - (2(p \cdot q) - (p \cdot p)) = (p \cdot p) + (q \cdot q) - 2(p \cdot q)$$

Therefore for a point $q \in R^d$ we can define $C(q)$ as the set of sites in S maximizing $h_p(q)$.

For each hyperplane h_p with $p \in S$ consider the halfspace above the hyperplane, i.e. $h_p^+ : x_{d+1} \geq 2(p \cdot x) - (p \cdot p)$. The set of maxima is described by the boundary of $\bigcap_{p \in S} h_p^+$. In other words the facial structure of the intersection polyhedron corresponds 1-1 to the Voronoi diagram $Vor(S)$ in R^d.

Thus note that an algorithm for computing intersection of halfspaces in R^{d+1} suffices to compute the Voronoi Diagram in dimension R^d.

9 Conclusion

We have shown a reduction of **Convex hull** to **Intersection of halfspaces** via Polarity.

A reduction of **Voronoi Diagram** to **Intersection of halfspaces** via Lifting onto the unit paraboloid.

A reduction of **Delaunay Triangulation** to **Voronoi Diagram** via duality of the incidence graph.

A single Convex Hull algorithm is the engine for solving all of these problems.

References

[Ede87] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Heidelberg, 1987.