Lesson 4. Intersection of Halfplanes and Halfspaces via an Incremental Algorithm

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January 26, 2004

1 Disclaimer

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2 Introduction

We do separately the 2D case and the 3D case. We add a proof of Euler's theorem for planar maps. Looking also at the original sources.

The proofs in [Mul93, pages 96-103] and [MR95, pages 341-245] are almost identical.

The original proof in [CS89] is quite different since there is no use of backward analysis. This proof does not appears to be in the backward analysis paper by Seidel [Sei92], however a similar result is derived via different techniques. In [Mat95] there is a somewhat different proof, this too is rather complex. A simpler proof appeared in [CMS92] and this one we show that is even simpler appeared for the first time in Mulmuley's book.

3 Variations from the 2D to the 3D cases

1. In 3D cutting a convex polyhedron with an halfspace produces many new vertices (a linear number in the number of facets in the worst case).

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2. Navigation to find all vertices in conflict with the new halfspace is done on the 1-skeleton of $H(S_j)$.

3. The expected structural cost is $O(1)$ because $H(S_j)$ has j facets at most and the number of vertices and edges is linear in the number of facets by Euler planar maps theorem.

   In summary, the expected number of new vertices is bounded by $(d/j)|H(S_j)|$.

4. Also we use a simplicity assumption that at most d halfsapces are incident to a vertex, where d is the dimension of the ambient space.

5.

4 Proving Euler Relation

Proof from [Ede87, page 242-243].

Some deifinitions. A graph $G(V, E)$ is called planar if there exists a mapping $m : V \to R^2$, and for every edge $e = (u, v) \in E$ there is a continuous mappings $f_e : [0, 1] \to R^2$ such that (1) $f_e(0) = m(u)$, $f_e(1) = m(v)$ and no two arcs meet except at endpoints. The embedding of a planar graph is a planar map. The images under m of V is the set of vertices, the co-domains of f_e are called arcs and the connected components of the complement of the union of the arcs are called facets.

We have a spherical map if the embedding is in S_2 the two dimensional sphere in R^3.

A straight line planar map (SLPM) is a planar map in which moreover each arc is a line segment.

Every planar graph admits a straight line planar embedding.

The boundary facial graph of a bounded convex 3D polyhedron is in 1-1 correspondence with a spherical map in such way to preserve dimensionality of the corresponding faces and incidence relations.

It is possbile to give an incidence preserving map from a spherical map to a planar map (just make a polar projection from a point interior to a facet).

So it suffices to prove Euler relation for Planar Graphs and it will hold for SLPM and for boundaries of convex 3D polyhedra.

Theorem 1 *Let v be the number of vertices, e the number of edges and f the number of facets of a planar map which is the embedding of a connected planar graph where no vertex is at infinity. Then the following holds*

(i) $v - e + f = 2$;

(ii) $e \leq 3v - 6$ *when* $v \geq 3$;

(iii) $f \leq 2v - 4$ *when* $v \geq 3$;

(iv) $f \leq (2/3)e$.

Proof of (i). By induction on the number of edges.

Let $\{e_1, ..., e_e\}$ be an enumeration of the edges in E and for each integer i, $1 \leq i \leq e$, $E_i = \{e_1, ..e_i\}$. Let V_i be the set of vertices incident to edges in E_i. Consider the sequence of graphs $G_i(V_i, E_i)$. We can choose the enumeration in such way that G_i is connected and for $i \leq v-1$, G_i is a tree.

The proof is by induction. Clearly for $i \leq v$, $|V_i| = |E_i| + 1$ and $|F_i| = 1$, thus it is true that $|V_i| - |E_i| + |F_i| = 2$. Thus assume that it is true for graph G_i we show it is true for graph G_{i+1}, where $i \geq v-1$.

Because all vertices have been embedded: $|V_{i+1}| = |V_i|$. The number of edges is increased by one: $|E_{i+1}| = |E_i| + 1$, and the new edge splits a face in two parts: $|F_{i+1}| = |F_i| + 1$. Now $|V_i| - |E_i| + |F_i| = 2$ implies $|V_i| - |E_i| - 1 + |F_i| + 1 = 2$ so: $|V_{i+1}| - |E_{i+1}| + |F_{i+1}| = 2$. Thus it is true for $i = n$, therefore $v - e + f = 2$.

(ii). Build from G_n a planar graph $G'(V', E')$ obtained by adding as many edges as possible while keeping the planarity property. Clearly $v = |V'|$, $e \leq |E'|$ and $f \leq |F'|$. In G' every facet is bounded by 3 edges since otherwise we could insert a new arc that satisfies planarity. The number of facet-edge incidences is $2|E'|$ since each arc is incident to two facets, and equal to $3|F'|$ since each facet is incident to 3 edges. Therefore $2|E'| = 3|F'|$. Plugging this in (i) yields (ii) and (iii) with equality for G' which implies (ii) and (iii) for G.

(iv). Is proved directly in G by noting that each facet in G is incident to at least 3 edges so $3f$ is at most the number of incidences $2e$.

Theorem 2 *Moreover, with the additional condition that each vertex is adjacent to at least 3 edges:*

(v) $e \leq 3f - 6$*;*

(vi) $v \leq 2f - 4$*;*

(vii) $v \leq (2/3)e$.

We build the dual graph G' of G by picking a vertex V' for every facet F of G, and joining by an arc facets in G sharing an arc. Thus we have a planar graph G' with $e' = e$, $f' = v$ and $v' = f$. The condition on the incidence of vertices is needed to avoid multi arcs in the dual. Thus (v),(vi) and (vii) follow by duality.

References

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