Computational Aspects of Game Theory **Bertinoro Spring School 2011**

Lecture 7: Repeated Games

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The theory of repeated games is suitable to analyze long-term strategic interactions among the same set of players. It explains phenomena that might at first seem irrational, like cooperation, altruism, revenge, and threats. One of the main messages that the theory conveys is that *repetition enables cooperation.* The fundamental insight is that "repetition acts as an enforcement mechanism, which enables the emergence of cooperative outcomes in equilibrium - when everybody is acting in his own best interests" (Aumann).

7.1 Definitions

Let $G(N, A, u)$ be a non-cooperative game, where N is the number of players, $A = A_1 \times A_2 \times \ldots \times A_N$ is the set of *strategy profiles*, and $u : A \to R^N$ is the payoff function.

For any game $G(N, A, u)$, we denote the corresponding *infinitely repeated game* by G^∞ (G^∞ is often called the *supergame*). In G^∞ the players $1, 2, \ldots, N$ play G infinitely many times in sequence. We say that G is the *stage game* of G^∞. In each period $t = 0, 1, 2, \ldots$ of play, player i chooses a strategy $a_i^t \in A_i$, and gets the payoff $u_i(a^t)$.

The choice of strategies in each of the first t periods is called a *history*, and is denoted by h^t. Formally, $h^t = (a^0, a^1, \ldots, a^{t-1}) \in A^t$. The infinite game play is described by $h^\infty = (a^0, a^1, \ldots)$.

Let $A^* = \cup_{k=0}^{\infty} A^k$. A *pure strategy* for player i in G^∞ is given by a function $s_i : A^* \to A_i$. The value $s_i(h^t) \in A_i$ will determine the strategy chosen by player i after t periods of play. Therefore player i makes his decisions based on the history of length t.

Similarly we can defined a mixed strategy in G^∞ as a function that determines the probability distribution according to which each player chooses a strategy after a given history of length t.

Example 7.1 (Strategies in G^∞) *Two examples of pure strategies in an infinitely repeated game G^∞ between two-players are* tit-for-tat *and* trigger*.*

Assume that "cooperate" (resp., "defect") indicates a pure strategy in the game G, which leads to a cooperative outcome (resp., an inferior outcome *which is a NE). This is what happens, e.g., in the prisoner's dilemma.*

In tit-for-tat, the player starts cooperating. If the other player defected, then she defects in the next round. Afterwards she resumes cooperation.

In trigger, the player starts cooperating. If the other player ever defects, then she defects forever.

The evaluation of payoffs in G^∞ can be done in several ways. We present two standard definitions[1]. Given an infinite sequence $p_0, p_1, \ldots$ of payoffs to player i,

- the *average reward* of i is $u_i = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{p_j}{n}$.

[1]There are technical problems with the definitions below because the limits do not need to exist. We do not consider these problems here.

- the *discounted reward* of i is $u_i = \sum_{j=0}^{\infty} \delta^j p_j$, where $0 \le \delta \le 1$. δ is called the *discount factor*.

Definition 7.2 (NE in G^∞) *A NE of G^∞ is a vector of mixed strategies $s = (s_1, s_2, \ldots, s_N)$ such that, for each player i, and for each $\bar{s}_i : A^* \to A_i$, we have*

$$u_i((s_1, s_2, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_N)) \ge u_i((s_1, s_2, \ldots, s_{i-1}, \bar{s}_i, s_{i+1}, \ldots, s_N)).$$

7.2 The Folk Theorem

Definition 7.3 (Minimax payoff) *Consider a game $G(N, A, u)$. The minimax payoff of player i is $v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i)$.*

The minimax payoff to player i is the level of payoff below which i cannot be forced by the other players.

Definition 7.4 (Individually rational payoff) *The vector of payoffs $r_1, r_2, \ldots, r_N$ (which is usually called* payoff profile*), where r_i is the payoff to player i, is individually rational if $r_i \ge v_i$, for all i's.*

Definition 7.5 (Feasible payoff) *Given a game G, a profile of payoffs $r_1, r_2, \ldots, r_N$ is feasible if it is a convex combination of payoff profiles in G.*

Theorem 7.6 (The Folk Theorem) *The payoff vectors, obtained as average rewards, corresponding to NE points in G^∞ coincide with the individually rational feasible payoffs in G.*

While equilibrium outcomes are self-enforcing (once at an equilibrium point, no player wants to unilaterally deviate), cooperative outcomes need an outside mechanism to enforce them.

The Folk theorem relates the *cooperative behavior* in game G to the non-cooperative behavior in G^∞. The repetition by itself, with the possibilities to retaliate, becomes the enforcement mechanism which is needed to make a cooperative outcome *stick*.

Bibliographic notes

Chapter 8 in [5] gives a good introduction to repeated games. Another source is the review [6]. Reflections on the subject and its relations to other areas of game theory can be found in [1, 2, 3, 4]. The proof of the Folk theorem can be found in [5], pp. 144-145.

References

[1] R.J. Aumann, Acceptable Points in General Cooperative n-Person Games, in Contributions to the Theory of Games IV, Annals of Math. Study (1959).

[2] R.J. Aumann, Survey of Repeated Games, in Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern (1981).

[3] R.J. Aumann, M. Maschler, Repeated Games of Incomplete Information, MIT Press, 1995.

[4] R.J. Aumann, L.S. Shapley, Long-Term Competition: A game-theoretic analysis, in Gale et al, editors, Essays in Game Theory (1994).

[5] M. J. Osborne, A. Rubinstein. A Course in Game Theory. The MIT Press: Cambridge, MA, 1994.

[6] S. Sorin, On repeated games with complete information, Mathematics of Operations Research, 11:147160, 1986.