Lecture 6: The Shapley Value

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The Shapley value provides an *a priori evaluation* of the *position* of each player in a cooperative game, based on the *contribution* that each player can make to the different coalitions.

6.1 Introduction

Definition 6.1 (Value) *A value is a function f that assigns to each cooperative game (N, v) a vector of payoffs $f(N, v) = (f_1, f_2, \dots, f_n)$, where f_i is the payoff to player i, such that $\sum_i f_i = v(N)$.*

Definition 6.2 (Marginal Contribution) *Given a cooperative game (N, v), the marginal contribution of player i to any coalition S with $i \notin S$ is*

$$\Delta_i(S) = v(S \cup \{i\}) - v(S).$$

Given a cooperative game (N, v), consider a permutation π on the set of players. Let us imagine that the players appear to "collect" their payoff according to the ordering π. For each player i, let us denote by p^i_π the set of players preceding i in π, i.e., $p^i_\pi = \{j | \pi(i) > \pi(j)\}$. The marginal contribution of player i with respect to π is $\Delta_i(\pi) = v(p^i_\pi \cup \{i\}) - v(p^i_\pi)$.

Definition 6.3 (The Shapley value) *Let (N, v) be a cooperative game. Let Π be the set of all permutations of $\{1, 2, \dots, n\}$.*

The Shapley value is the assignment $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ defined as

$$\phi_i = \frac{1}{n!} \sum_{\pi \in \Pi} \Delta_i(\pi).$$

Given any "ordering" of the players, where each order is equally likely, the Shapley value ϕ_i measures the expected marginal contribution of player i over all orders to the set of players who precede her.

6.2 An axiomatic characterization of the Shapley Value

We now present an axiomatic characterization of the Shapley Value. We will impose on any value f of a coalitional game four natural axioms, and then show that there is a unique value satisfying them, i.e., the Shapley Value.

The Four Axioms

1. $\sum_i f_i = v(N)$. (**Efficiency**)
2. We say that players $i, j \in N$ are *symmetric* in game (N, v) if they make the same marginal contribution to any coalition. If players $i, j \in N$ are symmetric, then $f_i = f_j$. (**Symmetry**)
3. We say that a player is a *dummy player* if her marginal contribution to any coalition is zero. If player i is a dummy player then $f_i = 0$. (**Nullity**)
4. Given two games (N, v) and (N, w), we define the game $(N, v + w)$, by $(v + w)(S) = v(S) + w(S)$, $\forall S \subseteq N$. $f(v + w) = f(v) + f(w)$. (**Additivity**)

We now state without proof a remarkable result.

Theorem 6.4 (Shapley) *There is a unique value satisfying the efficiency, symmetry, nullity, and additivity axioms, and this value is the Shapley value.*

6.3 Core vs Shapley Value: Examples

In this section, we illustrate via several examples the differences between the Shapley value and the core.

The first two examples show that the Shapley value is always well defined, even when the core is empty.

Example 6.5 (Majority Game) *Consider the majority game where $N = \{1, 2, 3\}$, $v(S) = 1$ if $|S| \geq 2$, and $v(S) = 0$ if $|S| \leq 1$. We have seen in Lecture 1 that the core of this game is empty. For the Shapley value, just observe that the three players are symmetric to see that $\phi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

Recall from Lecture 2 that we can represent weighted majority games as $[q; w_1, \ldots, w_n]$, where q is the quota, and the w_i's are the weights of the players.

Example 6.6 (Weighted Voting Game) *Consider a weighted voting game defined by $[3; 2, 1, 1, 1]$. Player 1 is giving a marginal contribution of 1 half the time (and a marginal contribution of 0 for the other half), so $\phi_1 = \frac{1}{2}$. The other players are symmetric, and thus $\phi = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. On the other hand, it is easy to see that the core is empty, because in any assignment the three players who get the least have an incentive to deviate.*

We now present two examples where the core is non empty, but the Shapley value is outside the core.

Example 6.7 *Consider the coalitional game (N, v), where $N = \{1, 2, 3\}$, and $v(\cdot)$ is defined as follows: $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = \frac{1}{2}$, $v(\{2, 3\}) = \frac{4}{5}$, and $v(\{1, 2, 3\}) = 1$.*

Note that player 1 is the weakest player. The Shapley value of player 1 is $\frac{7}{30}$. The other two players receive $\frac{23}{60}$ each. This imputation is not in the core since the coalition $\{2, 3\}$ can do better alone ($v(\{2, 3\}) = \frac{4}{5} > \frac{23}{60} + \frac{23}{60}$). The core is non empty. For example, the imputation $\frac{1}{5}, \frac{2}{5}, \frac{2}{5}$ belongs to the core.

Example 6.8 *Consider the coalitional game (N, v), where $N = \{1, 2, 3\}$, and $v(\cdot)$ is defined as follows: $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 0$, and $v(\{1, 2, 3\}) = 1$. We have seen in Lecture 2 that the core of this game contains only the imputation $(1, 0, 0)$. The Shapley value is instead $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.*

On the other hand, in the weighted graph game presented in lecture 5, we have seen that whenever the core is nonempty (which happens if and only if the graph does not have a negative cut) then the assignment $x_i = \frac{1}{2}\Sigma_{i \neq j} w(i,j)$ is in the core. It is easy to see that this assignment is the Shapley value of the weighted graph game.

6.4 The Potential of the Shapley value

Assume that a cooperative game (N, v) describes the problem of allocating a resource (or costs, profits) among N players. An approach could be that of assigning to each player i her marginal contribution to the grand coalition, i.e., $x_i = v(N) - v(N \setminus \{i\})$. The problem with this allocation is that it might not be feasible (the sum of the marginal contributions could exceed $v(N)$) or, if feasible, not efficient (the sum of the marginal contributions does not need be equal to $v(N)$).

Let $\mathbf{G}$ denote the set of all cooperative games in characteristic form. Consider a function $P : \mathbf{G} \to R$, which associates a real number $P(N, v)$ to every game (N, v). The marginal contribution of player i to the game, w.r.t. P, is defined as $\Delta^i P(N, v) = P(N, v) - P(N \setminus \{i\}, v)$.

Definition 6.9 (Potential function) *A function $P : \mathbf{G} \to R$ is called a* potential function *if it satisfies the following two conditions:*

1. $P(\emptyset, v) = 0$,
2. $\sum_{i \in N} \Delta^i P(N, v) = v(N)$, *for all games* (N, v).

Thus the potential function makes it possible to implement the marginal contribution idea that we mentioned above.

Theorem 6.10 (Hart-Mas-Colell) *There exists a unique potential function P. For every game (N, v) the payoff vector x, where $x_i = \Delta^i P(N, v)$ coincides with the Shapley value.*

We can write a recursive formula to describe the potential:

$$\begin{aligned} P(N, v) &= \frac{1}{N}\left[v(N) + \sum_{i \in N} P(N \setminus \{i\}, v)\right] \\ P(\emptyset, v) &= 0 \end{aligned}$$

Note that the core associates to each coalitional game a (possibly empty) set of payoff vectors, the Shapley value associates to each coalitional game a single vector of payoffs, and the potential a single real number.

6.5 The Shapley-Shubik Power Index

We now consider weighted majority games, which are special coalitional games where a coalition S is either winning (in which case $v(S) = 1$) or losing ($v(S) = 0$).

These games are one of the most natural places to apply the Shapley value.

In the context of weighted majority games, player i's marginal contribution to a coalition S is 1 if by joining S she can turn S from a losing coalition into a winning coalition. When this happens we say that player i is *pivotal* to coalition S.

Therefore the Shapley value (which in this special case has been called the *Shapley-Shubik power index*) assigns to each player the probability of being pivotal with respect to her predecessors, when the permutations are taken uniformly at random.

Definition 6.11 (Shapley-Shubik power index) *Consider the set*

$$T_i = \{\pi \in \Pi | v(p^i_\pi \cup \{i\}) = 1 \text{ and } v(p^i_\pi) = 0\}.$$

The Shapley-Shubik power index assigns to player i the value $\phi_i = \frac{|T_i|}{n!}$.

Bibliographic notes

The Shapley value has been introduced by Shapley in 1953 [9]. Examples, applications, and reflections on the Shapley value can be found in [1, 2, 3, 4].

The potential of the Shapley value has been introduced by Hart and Mas-Colell [7, 8].

The application of the Shapley value to the scenario of weighted majority games is analyzed in [10]. The Shapley-Shubik index is far from being the only index used to analyze the distribution of power in weighted majority games. One of the most used power indices is Banzhaf index [5]. A general reference on power indices and voting power is [6].

References

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