Computational Aspects of Game Theory **Bertinoro Spring School 2011**

Lecture 5: The Core of Cooperative Games

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The core is by far the cooperative solution concept most used in Economic Theory. In this lecture, we first present a characterization of the nonemptiness of the core, and then we examine the core of some important coalitional games.

5.1 Introduction

A cooperative game (or coalitional game) in *characteristic form* is a pair (N, v), where N is a finite set (the a set of players) and a v is a real-valued function $v : 2^N \to R$ that associates to each coalition $S \subseteq N$ a value $v(S)$ which represents the total payoff available to players in S. It is required that v assigns zero to the empty set.

Therefore in a cooperative game the focus is on what coalitions of players can achieve rather than on strategies to be chosen by individual players.

Players act efficiently when they form a single coalition which includes all of them, the *grand coalition.* The main goal of a cooperative game is to find *reasonable* distributions of the payoff of the grand coalition.

Accordingly, the notion of stability has to take into account the incentive to unilaterally deviate that a coalition of players, rather than just individual players, might have from any proposed distribution of the total payoff.

This is captured by the core, which is thus the counterpart of the NE in the context of coalitional games.

Definition 5.1 (Imputation). *Given a cooperative game (N, v), an imputation $x = (x_1, x_2, \ldots, x_n)$ is a distribution of the total payoff among the players such that*

1. $x_i \geq v(\{i\}), \forall i$, *i.e., a player receives at least as much as she could obtain on her own, without cooperating with anyone else (*individual rationality*).*
2. $\sum_{i=1}^n x_i = v(N)$ *(*efficiency*).*

Definition 5.2 (Core). *The core is the set of imputations x such that $x(S) = \sum_{i \in S} x_i \geq v(S)$, for all $S \subseteq N$, with equality for $S = N$.*

An important distinction is between cooperative games with *transferable utilities* (TU games), and cooperative games without transferable utilities (NTU games). In TU games, such as the cooperative games that we have defined above, we do not make any assumption about how the members of a given coalition S divide the payoff $v(S)$. On the contrary, in NTU games there is no transfer of *utility* between players, and the "worth" of a coalition S cannot be just specified by $v(S)$. In the following, we will deal only with TU games.

5.2 A characterization of the nonemptiness of the core

Since the core might be empty, it is important to introduce analytical tools to study the question of its possible emptiness. The Bondareva-Shapley Theorem below provides the required characterization.

Definition 5.3. *Let (N, v) be a coalitional game. A set of weights $w(S)$, where $0 \leq w(S) \leq 1$, for all $S \subseteq N$, is a* balanced sequence *(or balancing set of weights) if $\forall i \in N$*

$$\sum_{S, i \in S} w(S) = 1.$$

We can think of these weights as the fraction of time that each player devotes to each coalition he is a member of. Note that, for a given coalition, this fraction is the same for all the players in the coalition.

Definition 5.4. *A coalitional game (N, v) is balanced if and only if, for every balancing set of weights w, we have*

$$\sum_{\emptyset \neq S \subseteq N} w(S) v(S) \leq v(N).$$

In a balanced game, any "time-sharing" arrangement feasible for the different coalitions is feasible for the grand coalition as well.

Example 5.5 (Balanced sequences for the majority game)**.** *Let us look at the notion of balanced sequence and balanced game in the framework of a simple coalitional game, the majority game, where $N = 1, 2, 3$, and*

$$\nu(S) = \begin{cases} 1 & |S| \geq 2 \\ 0 & |S| \leq 1 \end{cases}$$

Consider the following 3×7 matrix $F = (f_{ij})$

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
1	1	0	0	1	1	0	1
2	0	1	0	1	0	1	1
3	0	0	1	0	1	1	1
	$w_{\{1\}}$	$w_{\{2\}}$	$w_{\{3\}}$	$w_{\{1,2\}}$	$w_{\{1,3\}}$	$w_{\{2,3\}}$	$w_{\{1,2,3\}}$

whose rows are indexed by the players and whose columns are indexed by the coalitions. To each column we associate an element of a sequence w.

The (i, j)-th entry of the matrix F is equal to 1 if the i-th player belongs to the set indexed by the j-th column. For a sequence to be balanced, we must have $\sum_j f_{ij} w_j = 1$, $i = 1, 2, 3$.

It is easy to see that the sequence $(0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ is balanced. Indeed each player (matrix row) is involved with exactly two of the nonzero elements of the sequence, and we obtain, for each row, $\frac{1}{2} + \frac{1}{2} = 1$.

On the other hand

$$w_{\{1,2\}} + w_{\{1,3\}} + w_{\{2,3\}} = \frac{3}{2} > v(N),$$

so that the majority game is not balanced.

We can now state the theorem which characterizes the non-emptiness of the core.

Theorem 5.6 (Bondareva-Shapley)**.** *The coalitional game (N, v) has a non-empty core if and only if it is balanced.*

The Bondareva-Shapley's Theorem is usually proved using duality techniques in Linear Programming, which take advantage of the definition of the core in terms of linear inequalities. Instead of doing this, we use a zero-sum non-cooperative game which can be generated from any cooperative game. By using the minimax theorem (see Lecture 4), we prove a correspondence between the NE of the non-cooperative game and the core imputations (Aumann's Theorem). This correspondence then implies the Bondareva-Shapley's Theorem.

Our approach to the proof of this theorem allows us to show a connection between cooperative and non-cooperative games which is of independent interest.

We now show how to derive, from any coalitional game (N, v), a two-player non-cooperative zero-sum game, which we will denote by $G(N, v)$. We then examine the relationship between the core of (N, v) and the NE of $G(N, v)$.

$G(N, v)$ is defined as follows:

- The set of strategies available to the row player is $N = \{1, \dots, n\}$, the set of players in the coalition game (N, v).
- The set of strategies of the column player is the set of coalitions formed by at least one player.
- The payoff matrix of the row player is the $n \times (2^n - 1)$ matrix $A = (a_{ij})$, where:

$$a_{ij} = \begin{cases} \frac{v(N)}{v(S_j)} & \text{if } i \in S_j \\ 0 & \text{otherwise} \end{cases}$$

The game is a zero sum game, so that the column player's payoff matrix is $-A$.

Note that if $v(S_j) = 0$, then the entry a_{ij} is undefined. In this case, we can assign to a_{ij} a very large number. Since the game is zero sum, the column player will never choose the column j. This is equivalent to restricting the strategies of the column player to the subsets S_j such that $v(S_j) > 0$.

The row player gets a positive payoff if and only if she chooses a "player" belonging to the coalition chosen by the column player. The amount received by the row player is inversely proportional to the worth of the coalition chosen by the column player.

Let (s, t) be a NE for $G(N, v)$. We denote by $w\,(G(N, v))$, or w for short, the payoff of the row player when the NE (s, t) is played, i.e., $w = s^T At$. Because of the zero-sum property, w is the *value* of the game, and it is always well defined.

We begin by proving the following easy lemma.

Lemma 5.7. $0 < w\,(G\,(N, v)) \leq 1$.

Proof. We first show that $w > 0$. Note that all the columns of A have at least one nonzero entry. The row player can play $x = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, thus making the components of the row vector $x^T A$ all positive. Therefore w will be greater than zero.

To show $w \leq 1$, we just observe that the column player can always prevent the row player from getting a payoff greater than one by playing the last column, whose entries are all ones. □

Theorem 5.8 (Aumann)**.** *Assume that the game (N, v) is balanced. Then $w(G(N, v)) \geq 1$.*

Proof. Assume for simplicity that the game (N, v) is $0 - 1$ normalized, i.e., that $v(\{i\}) = 0$ and $v(N) = 1$.

Assume by contradiction that $w(G(N,v)) = t$, $0 < t < 1$. Then there is a mixed strategy y for the column player that guarantees that the payoff to the row player will be at most t. Let $y_S > 0$, for each coalition S in a family F with $v(S) > 0$, for all $S \in F$. We have $\sum_{S\in F; i\in S} \frac{y_S}{v(S)} \le t$, for all rows i. Thus $\sum_{S\in F; i\in S} \frac{y_S}{tv(S)} \le 1$.

We now define $w(S) = \frac{y_S}{tv(S)}$ for all $S \in F$. Then $\sum_{S\in F; i\in S} w(S) \le 1$. Now define $w_i = 1 - \sum_{S\in F; i\in S} w(S)$. Consider now the family of coalitions T consisting of F and all the $\{i\}$'s.

Then $\sum_{S\in T; i\in S} w(S) = \sum_{S\in F; i\in S} w(S) + w_i = 1$, which shows that T is a balanced family with balancing weights $\{w(S)\}$. Therefore by the assumption that the game is balanced we have $\sum_{S\in T} w(S)v(S) \le v(N)$, from which, since $v(\{i\}) = 0$, we obtain $\sum_{S\in F} w(S)v(S) \le v(N)$. But this implies $\sum_{S\in F} \frac{y_S}{t} \le v(N) = 1$, i.e., $\sum_{S\in F} y_S \le t < 1$, which contradicts the fact that y is a mixed strategy for the column player. Therefore we must have $w(G(N,v)) \ge 1$. □

We can now easily prove the Bondareva-Shapley's Theorem.

Proof. (of Theorem 5.6). We first show that if the core is non empty then, for every balancing set of weights w, we have $\sum_{\emptyset \ne S \subseteq N} w(S)v(S) \le v(N)$. Let x be in the core. Then $\sum_i x_i = v(N)$, and $\sum_{i\in S} x_i \ge v(S)$, for all $S \subseteq N$. Consider a balancing set of weights $\{w(S)\}$. We have $w(S)\sum_{i\in S} x_i \ge w(S)v(S)$, from which $\sum_S w(S)\sum_{i\in S} x_i \ge \sum_S w(S)v(S)$.

Now we have $\sum_S w(S)\sum_{i\in S} x_i = \sum_i x_i = v(N)$, and thus $\sum_S w(S)v(S) \le v(N)$.

For the other direction, we use Theorem 5.8. Assume that $w(G(N,v)) \ge 1$. Then there is a mixed strategy x for the row player which gives a payoff at least one, independently of the pure strategy chosen by the column player. Then, for all S, we have

$$\sum_{i\in S} x_i \frac{v(N)}{v(S)} \ge 1,$$

i.e., $\sum_{i\in S} x_i v(N) \ge v(S)$. Therefore the vector $xv(N)$ is in the core of (N,v).

□

5.3 The core of the weighted graph game

Let us consider an undirected graph $G = (V,E)$, with an integer weight $w(i,j)$ on each edge i,j. To the graph G, we associate the coalitional game (N,v), which we call weighted graph game (wGG), where N, the set of players, is the set of vertices of G, and $v(\cdot)$ is defined as

$$\forall S \subseteq V, \qquad v(S) = \sum_{i,j\in S} w(i,j).$$

We can think of the nodes of G as cities, of the edges as highways, and of the weights as profits (or losses) from the highways. Then the problem is to distribute "fairly" among the cities the income generated by the network of highways.

Consider the imputation $x_i = \frac{1}{2}\Sigma_{i\ne j} w(i,j)$. We want to see if such imputation is in the core. In order for this to hold true, we must have $\forall\, S \subseteq N$, $x(S) = \sum_{i\in S} x_i \ge v(S)$. For any coalition S, we must have $x(S) + x(N-S) = x(N) = v(N)$. If x is in the core, then both $x(S) \ge v(S)$, and $x(N-S) \ge v(N-S)$ must hold true. If we consider the definition of x, we see that the union of the subgraphs built on S and $N - S$

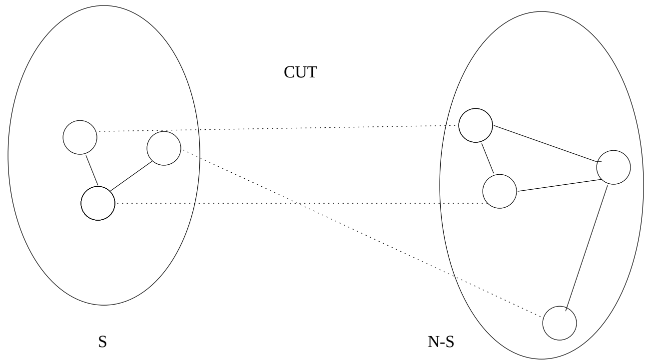

Figure 5.1: Sets S and $N - S$ and the cut in G. Edges in the cut are represented by the dotted lines.

lacks the edges which connect S to $N - S$. Those edges form a cut in G (see Figure 5.1). It is easy to see that we can simultaneously satisfy

$$x(N) = v(N)\,,\ \forall S\,,\, x(S) + x(N - S) = x(N) = v(N),\, x(S) \geq v(S),$$

if and only if there is no set S such that the sum of the weights on the edges separating S from the rest of the graph is negative. Otherwise we would have $v(S) + v(N - S) > v(N)$.

So the imputation $x_i = \frac{1}{2}\Sigma_{i \neq j} w(i, j)$ is in the core if and only if there is no negative cut in G[1].

We now assume that there is a negative cut in G. We can then choose S so that the graphs induced by S and $N - S$ are separated by a negative cut. Then $v(S) + v(N - S) > v(N)$; on the other hand, for any imputation x we must have $x(N) = x(S) + x(N - S) \geq v(S) + v(N - S) > v(N)$. But $x(N) = v(N)$, by definition of imputation. Therefore the core is empty.

The arguments just given above prove the following result.

Lemma 5.9. *The core of the wGG game is nonempty if and only if there is no negative cut in G.*

5.4 The core of the minimum spanning tree game

We now analyze the problem of distributing the costs induced by the connection of many customers to a common resource. This problem appears in a variety of contexts, and in particular can be used to model the problem of sharing costs in multicasting transmissions.

Let N be a nonempty finite set of customers, and $0 \notin N$ be a supplier. Let $N^* = N \cup \{0\}$, and c_{ij} be the cost of connecting i and j, $i, j \in N^*$, by edge e_{ij}.

Let $S \subseteq N$. A minimum cost spanning tree T_S is a tree with vertex set $S \cup \{0\}$, and a set of edges E_S that connect the members of S to the supply 0 so that the total cost of the connections is minimized. (Figure 5.2 shows a graph and its minimum cost spanning tree.)

For all $S \subseteq N$, we define the cost function $c(S) = \sum_{(i,j) \in E_S} c_{ij}$.

The minimum cost spanning tree game (MCST) is defined by the pair (N, C), where $C = (c_{ij})$ is the matrix of

[1]This imputation turns out to be the well know Shapley value of the game. The Shapley value is an imputation intended to reflect the marginal contribution of each player to the outcome, averaged over all possible "orders of arrival" of the players. We will deal with the Shapley value in lecture 6.

costs. Note that in the MCST game we minimize costs, rather than maximize payoffs, so that an imputation in the core has to satisfy $\sum_{i \in S} x_i \leq c(S)$, for all $S \subset N$.

Theorem 5.10. *The core of the MCST game is nonempty.*

Proof. Let T_N be a MCST for the complete graph with vertex set $N^* = N \cup \{0\}$, and costs $C = (c_{ij})$. Let E_N be the edge set of T_N. Let $i \in N$. There is a unique path $(0, i_1, \ldots, i_t, i)$ in T_N connecting i to 0. Let us consider the imputation x_i that assigns to i the cost of the edge $e_{i_t,i}$. We claim that this imputation is in the core. First of all, by construction, we clearly have $x(N) = \sum_i x_i = c(N)$. We now need to show that $x(S) \leq c(S)$, for all $S \subseteq N$. Consider a MCST T_S with vertex set $S \cup \{0\}$, and edge set E_S. Now expand T_S to a graph H_N with vertex set N^* and edge set $E_N^{'}$, by adding, for each vertex $i \in N \setminus S$ the edge $(j, i) \in E_N$, which is on the path from 0 to i. Note that the graph H_N is connected and has N edges. Therefore it is a tree. We then have

$$c(S) + x(N \setminus S) = \sum_{e_{ij} \in E_N^{'}} c_{ij} \geq \sum_{e_{ij} \in E_N} c_{ij} = c(N) = x(N),$$

which implies that $c(S) \geq x(S)$. □

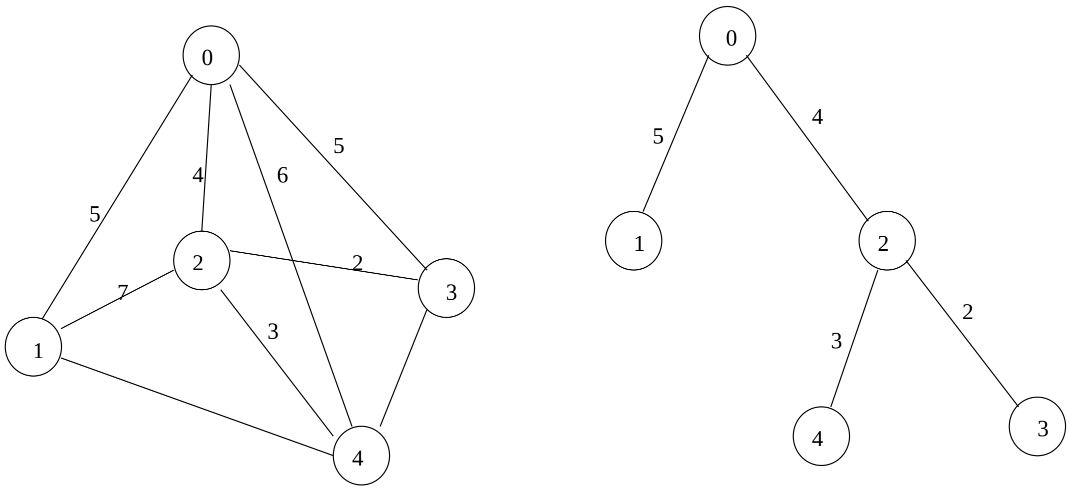

Figure 5.2: A graph and its MCST

There are some problems with the actual “fairness” of the core allocation introduced in the proof of Theorem 5.10.

1. It gives an unfair advantage to the leaves of the tree, and a corresponding disadvantage to the nodes close to the resource. For instance, the nodes directly connected to the resource in the MCST have to pay in full the cost of their link to 0.

2. It does not satisfy “cost monotonicity”. This fact is best illustrated by an example. In Fig. 5.3 we see that the amount payed by player 2 in the MCST for the graph at the top of the picture is 3, while for the graph at the bottom such cost is 3.5. This happens despite the fact that the only difference between the two graphs is that in the one at the bottom player 2 has a smaller cost of connection to 0 (and all the other edge costs remain the same).

These arguments have lead to the quest for “fairer” core allocations for the MCST game.

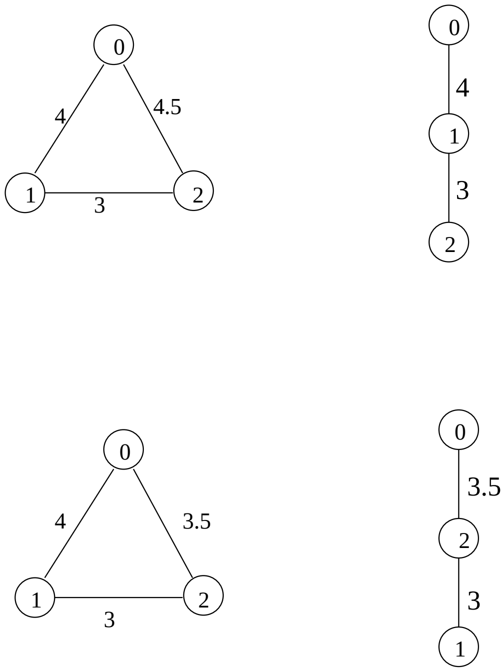

Figure 5.3: An example that illustrates cost non-monotonicity.

5.5 Market games

We now consider an economic application of the concept of core.

Definition 5.11. *A* market *is a 4-tuple* (T, G, A, U)*, where*

- T *is the set of* traders
- G *is the* commodity space*, typically the non-negative orthant of a real vector space.*
- A *is a set of* initial endowments. $A = \{a^i : a^i \in G, i \in T\}$.
- U *is a set of* utility functions. $u^i : G \to \mathbf{R}, i \in T$. *We assume that* u^i *is a continuous and concave function.*

For any $S \subseteq T$*, an* S*-feasible allocation is a set* $x^S = \{x^i : i \in S\} \subseteq G$ *such that* $\sum_{i \in S} x^i = \sum_{i \in S} a^i$.

We now derive from the market (T, G, A, U) a coalitional game (N, v), which we will call *market game*. The market game (N, v) is defined by taking $N = T$, and

$$v(S) = \max_{x^S} \sum_{i \in S} \sum_{i \in S} u^i(x^i),$$

where the maximum is over all S-feasible allocations x^S.

Definition 5.12 (Totally balanced game)**.** *A subgame of* (N, v) *is a game* (R, v)*, where* $R \subseteq N$*, and* v *is restricted to the subsets of* R*. A game* (N, v) *is totally balanced if all its subgames are balanced.*

Not only the core of a totally balanced game is nonempty, but also all its subgames have a nonempty core.

Theorem 5.13 (Shapley-Shubik)**.** *A coalitional game is a market game if and only if it is totally balanced.*

Bibliographic notes

The book [8] provides a complete presentation of the theory of cooperative games. Chapter 3 is dedicated to the core of TU games.

The notion of core has been defined by Gillies in 1953 [5]. Bondareva-Shapley Theorem has been proved independently by Bondareva [2] in 1963, and by Shapley [9] in 1967. The proof that we have presented is based on a proof by Aumann [1].

The weighted graph game of Section 5.3 has been introduced by Deng and Papadimitriou in [3]. The minimum spanning tree game has been studied in [6]. The proof of the characterization of market games in terms of totally balanced games can be found in [10].

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