Lecture 3: Fair Division

Lecturer: Bruno Codenotti

Fair division problems are concerned with finding ways of dividing a resource among several parties in such a way that all recipients *believe* that they have received a *fair* amount, according to some notion of fairness. The problem is difficult because each recipient may have a different measure of the value of the resource. Examples of problems of fair division are dividing up the property in an estate, determining the borders in an international dispute, etc.

The questions in the area of fair division range from the existence problem (does a division of the goods with the specific fairness property exist?) to the optimization problem of actually finding either an exact or an approximate fair subdivision. We present some of the basic results in this field.

3.1 Fairness: Basic Definitions

The notion of fairness can come in several different flavors. Here we consider envy-freeness and proportionality.

Definition 3.1 (Envy). *A player who would prefer the goods assigned to another player to her own goods is said to envy the other player.*

Definition 3.2 (Envy-free Allocation). *An allocation of goods to players is called an envy-free allocation if no player envies any other player.*

Definition 3.3 (Proportional Allocation). *An allocation of goods to n players is called a proportional allocation if each player, based on her own preferences, evaluates the piece she received worth at least $\frac{1}{n}$-th of the whole amount of goods.*

Note that an envy-free allocation must be proportional, whereas a proportional allocation need not be envy-free. However, for $n = 2$, the two notions are clearly equivalent.

3.2 Cake Cutting: Proportional Allocations

The most popular example of fair division is perhaps the *cake-cutting* problem. This basic question has been present in human history since ancient times. It evolved into a rich discipline which exemplifies the role of mathematics in the social sciences.

In the simplest instance of the cake-cutting problem we are given two people who must share a homogeneous cake, like in the Fair Division Example of Lecture 2. As we have seen, the procedure we can adopt corresponds to a very old custom, according to which one of the two people will cut the cake, and the second one will choose a piece (the divide-and-choose procedure). The optimal strategy for the first player is to divide the cake into two equal parts, while the optimal strategy for the second player is to choose the bigger piece. If both players adopt their optimal strategies, then each will get half the cake.

What about a more general scenario, where the two players have different tastes (which they keep secret) and where the cake is heterogeneous, so that some pieces of the cake might be more attractive not only because of their size, but also because they contain special items, or are richer than other pieces of a certain ingredient? In this case, the objective of a procedure of fair division will be to provide each player with a piece of the cake whose value, according to such player's privately known preferences, is at least half the total *value* of the cake. Does the divide-and-choose procedure still give a fair outcome?

The first player will cut the cake in two pieces which she considers (according to her own preferences) equivalent; the second one will choose the piece which he considers more valuable. No matter what the second player does, the first one will be left with a piece she considers a fair share. Indeed she had divided the cake into pieces which looked equivalent to her. The second player chooses a part he considers at least as valuable as the one he leaves to the other player. So we get a division which is envy-free (no one thinks the other one got a better piece), and thus also proportional (both players think they have received at least a half the value of the cake). As already mentioned, in the two player case these two notions are equivalent: if one thinks he has received at least half the value of the cake, he also must think that the other player cannot have received more than half the value, and so he does not envy her.

The problem becomes more interesting when there are n-players, for $n > 2$. In this case, the equivalence between envy-free division and proportionality does not hold. A player might think he has received a piece whose value is $\frac{1}{n}$-th of the total value (so that proportionality is respected), but he can also think that some other player received more than $\frac{1}{n}$-th of the total value, and he will envy her.

We now describe a procedure, known as *Banach-Knaster last-diminisher procedure*, which provides a proportional division of a cake among n people.

1. Player 1 cuts a piece of cake (that she considers to represent $\frac{1}{n}$-th of the total value).
2. That piece is passed around the players. Each player either lets it pass (if she considers it at most $\frac{1}{n}$-th of the total value) or trims it down further (to what she considers worth $\frac{1}{n}$-th of the total value).
3. After the piece has made the full round, the last player to cut something off (the *last diminisher*) has to take it.
4. The rest of the cake (including the trimmings) is then divided among the remaining $n - 1$ players. Once down to $n = 2$, the two remaining players play the divide-and-choose two-player game.

It is easy to see that this procedure guarantees a proportional division, although it does not provide an envy-free division.

A player will receive a piece which she was the last to trim. She trimmed it so that it became (for her) worth $\frac{1}{n}$-th of the whole. Therefore she knows that she received her share (proportionality). But is it true that nobody is going to receive more (in her eyes)? The answer is no. All the players (except for the last two to choose) can envy players who pick their pieces later; having exited the game, they can no longer prevent the other players from getting pieces they regard worth more than $\frac{1}{n}$-th of the total value. It is easy to see that this cannot happen for the last two players.

We now give a sketch of another algorithm, based on Hall's Theorem, which attains a proportional subdivision of a cake among n players.

We recall the statement of Hall's Theorem.

Theorem 3.4 (Hall's Theorem)**.** *Let $G = (V, E)$ be a bipartite graph, where $V = X \cup Y$. G contains a matching saturating all the vertices in X if and only if*

$$|N(S)| \geq |S|, \ \forall S \subseteq X\,,$$

where $N(S)$ denotes the set of neighbors of S. If $|X| = |Y|$, then G contains a perfect matching.

We are given a cake and n players $p_1, p_2, \ldots, p_n$. We let p_1 divide the cake into n pieces $x_1, x_2, \ldots, x_n$ which look equally desirable to her. We now construct a bipartite graph $G = (X \cup Y, E)$, where X is the set of pieces of cake and Y is the set of players. There is an edge in G between p_i and x_j if p_i values x_j at least $\frac{1}{n}$th of the cake. Note that p_1 is connected by an edge to all the x_j's. Therefore there exists a subset $S \subseteq X$ such that $|N(S)| \geq |S|$. It is now possible to prove the existence (and actually to construct) a smallest subset $T \subseteq X$ such that $|N(T)| = |T|$ and $|N(Q)| > |Q|$, for all proper nonempty $Q \subset T$. Hall's Theorem guarantees that we can find an assignment of the pieces in T to the players in $N(T)$. By construction, the players in $N(T)$ consider any of the pieces in T worth at least $\frac{1}{n}$th of the cake, while the remaining players are not interested in them, and are happy to start dividing what is left of the cake. We can then proceed by induction.

3.3 Sperner's Lemma

We will now prove Sperner's Lemma, a simple and important combinatorial result which will be the basic tool needed in the proof of existence of an envy-free division of a cake among n players.

Sperner's Lemma readily follows from a simple parity argument in graph theory.

Theorem 3.5. *Let $G = (V, E)$ be a graph, and let $d(v)$ denote the degree of $v \in V$. Then $\sum_{v \in V} d(v)$ is an even number.*

Proof. The quantity $\sum_{v \in V} d(v)$ counts the edges in G twice. □

Corollary 3.6. *Let $G = (V, E)$ be a graph. The number of vertices of odd degree in G is even.*

Proof. Let V_1, $V_2 \subseteq V$ denote the sets of vertices of odd and even degree, respectively. We have

$$\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v).$$

Since the quantities $\sum_{v \in V} d(v)$ and $\sum_{v \in V_2} d(v)$ are even, then the quantity $\sum_{v \in V_1} d(v)$ must also be even. But this is a sum of odd numbers, and thus it can be even if and only if the number of terms in the summation is even. Therefore $|V_1|$ is even. □

Before proving Sperner's Lemma, we need some definitions.

Definition 3.7 (Unit n-simplex)**.** *The unit n-simplex is the subset of $\mathbf{R^{n+1}}$ defined as*

$$\{(x_0, x_1, \ldots, x_n) \in \mathbf{R^{n+1}} | \sum_{i=0}^{n} x_i = 1,\, x_i \geq 0,\, \forall i\}.$$

Definition 3.8 (Simplicial Subdivision of Triangles)**.** *Let T be a closed triangle in the plane. A subdivision of T into a finite number of triangles is said to be simplicial if any two intersecting triangles share either a vertex or a side. The triangles obtained by a simplicial subdivision are called elementary triangles. The simplicial subdivision is also called triangulation.*

Example 3.9 (Barycentric Subdivision of the unit n-simplex)**.** *An important subdivision is the barycentric subdivision.*

The barycentric subdivision of a unit n-simplex S_n consists of $(n+1)!$ simplices. The vertices $v_0, v_1, \ldots, v_n$ of each elementary simplex can be associated with a permutation $p_0, p_1, \ldots, p_n$ of the vertices of S_n in such a way that v_i, $i = 0, 1, \ldots, n$, is the barycenter of the points $p_0, p_1, \ldots, p_i$, $i = 0, 1, \ldots, n$.

In particular, the barycentric subdivision of a line segment S_1 (unit 1-simplex) consists of two smaller segments, each connecting one endpoint of S_1 to the midpoint of S_1 .

The barycentric subdivision of a triangle S_2 (unit 2-simplex) divides it into six triangles; each elementary triangle has one vertex at the barycenter of S_2, another one at the midpoint of some side, and the last one at one of the original vertices.

Definition 3.10 (Proper labeling)**.** *Given a simplicial subdivision of a triangle T, a labeling of the vertices with the elements of the set $\{0, 1, 2\}$ is called proper if $\{v_0, v_1, v_2\}$, the three vertices of T, are labeled $\{0, 1, 2\}$, respectively, and, for all i, $0 \le i < j \le 2$, each vertex on the side of T which joins vertices i and j is labeled either i or j. No assumption is made on the labeling of the internal vertices, i.e., vertices placed in the interior of T.*

Definition 3.11 (Distinguished Triangle)**.** *A triangle labeled with all the three labels is called either distinguished triangle or completely labeled triangle.*

We are now ready to prove the two-dimensional version of Sperner's Lemma. The extension to the n-dimensional case is quite simple.

Lemma 3.12 (2-Dim Sperner)**.** *Every simplicial subdivision of a triangle T, with a proper labeling, contains an odd number of distinguished triangles.*

Proof. Let T_0 be the region outside T, and let $T_1, T_2, \cdots, T_n$ be the triangles in the subdivision. Let us consider a graph $G = (V, E)$ on the vertex set $V = \{v_0, v_1, \cdots, v_n\}$, where v_i corresponds to the triangle T_i. The edge set is defined as follows: $(v_i, v_j) \in E$ if the common boundary between T_i and T_j is a side whose endpoints are labeled 0 and 1.

First of all we show that v_0 has odd degree. Let s be the side of T whose endpoints are labeled 0 and 1. T_0 is adjacent to the triangles with one side on s and endpoints 0 and 1. It is easy to see that the number of these triangles is odd. Therefore v_0 has odd degree.

Corollary 3.6 implies that an odd number of the remaining vertices of G (which correspond to elementary triangles) have an odd degree. None of them can have degree three, for otherwise it would have three sides labeled 0 and 1, which is impossible. Therefore the vertices of odd degree have degree one. But a vertex v_i has degree one if an only if T_i is a distinguished triangle. This shows that there must be an odd number of distinguished triangles. □

3.4 Sperner's Lemma and Envy-Free Division

Let us be given a rectangular cake of horizontal length one. Consider a set of $n - 1$ vertical cuts which partition the cake into n rectangular portions. Let x_i be the horizontal length of the i-th (from the left) piece of cake.

We have

$$x_i \ge 0, \;\; i = 1, \ldots, n$$
$$x_1 + x_2 + \ldots + x_n = 1 \,.$$

The set S of possible partitions of the cake forms a unit $(n - 1)$-simplex. Each $s \in S$ is called a *cut-set.*

We make the following assumptions:

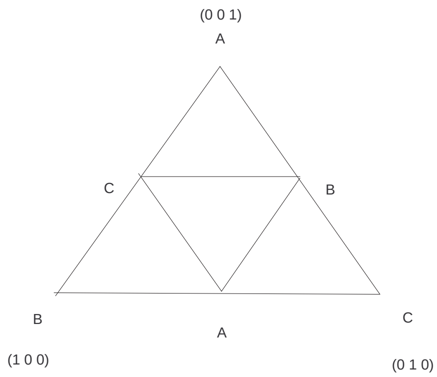

Figure 3.1: Labeling of the vertices according to the names of the owners.

1. All the players are *hungry*, i.e., any piece of cake is better than the empty piece;
2. The preference sets are *closed*. Let $\{t_1, t_2, \ldots\}$ be a sequence of cut-sets. Given $s \in S$, if s is preferred to t_i, $\forall\, i = 1, 2, \ldots$, then s is also preferred to $t = \lim_{n\to\infty} t_n$.

Theorem 3.13. *Under assumptions (1) and (2) above, there exists an envy-free cake division.*

Proof. We first prove the theorem for $n = 3$, and then give a sketch of the proof for the n-dimensional case.

Consider the unit 2-simplex

$$x_i \geq 0,\ \ i = 1, 2, 3,$$
$$x_1 + x_2 + x_3 = 1\,,$$

which represents the possible partitions of the cake into three parts, and a corresponding triangulation.

Let A, B, and C be the three players. Let us assign an *ownership* to each of the vertices of the elementary triangles, in such a way that each elementary triangle has all the three labels A, B, and C (see Figure 3.1). Notice that this kind of assignment (of labels to vertices) is always possible if we use the regular triangulation of Figure 3.1.

Let us now construct an auxiliary labeling from the set $\{1, 2, 3\}$ as follows. For each vertex of the subdivision, which corresponds to a cut-set, we "ask" the owner of the vertex to choose a piece. The answer will be an integer i from $\{1, 2, 3\}$. We label the vertex with such an i. We leave as an exercise the proof that this gives a proper labeling.

We can thus invoke Sperner's Lemma to state the existence of an elementary triangle with all the three labels, i.e., a distinguished triangle. If this triangle is small enough, then there are three very similar cut-sets for which the three players prefer different pieces. We can now use assumption (2) to turn this approximate result into an exact one.

The proof can be easily generalized to the n-dimensional case. The only difficulty lies in finding an initial assignment of ownerships such that all elementary simplices have all the n labels. The previous subdivision does not work. However one can use the *barycentric subdivision* (see Example 3.9, and Figure 3.2) which

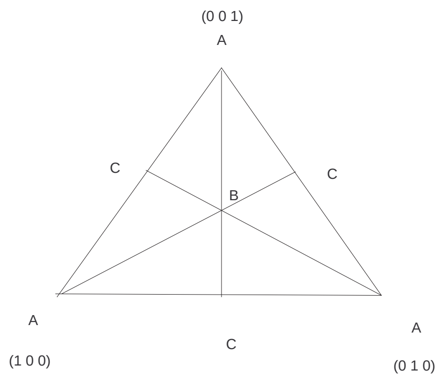

Figure 3.2: Barycentric subdivision of the unit simplex, and the associated labeling.

readily gives the required assignment. Given such an assignment, the proof is the same as in the case $n = 3$. □

Bibliographic notes

The problem of fair division has been introduced by Steinhaus in 1948. In [15] he reports his solution (for $n = 3$) and the generalization to arbitrary n by Banach and Knaster.

A discussion of the many real world applications of the divide-and-choose procedure can be found in [6].

The book by Brams and Taylor [4] is a very good introduction to the subject. It also offers a wide spectrum of applications for techniques used in fair division.

The description of the proportional allocation algorithm based on Hall's Theorem – which we have seen at the end of Section 3.2 – can be found in a book written by Robertson and Webb [11].

Sperner's Lemma was proved in [14]. We have followed the proof presented in [3].

Bondy and Murthy book presents the derivation of Sperner's Lemma from basic parity arguments in graph theory (p. 10, and pp. 21-23 in [2]).

The expository article [16] provides a gentle introduction to the role of Sperner's Lemma in proving the existence of approximate envy-free solutions to various fair division problems.

In this lecture, we have been focusing on proportional and envy-free allocations. There are other notions of fairness; for instance it might be worth mentioning the idea of a *consensus-halving* division, where each player believes that the allocations are all equal. [13] illustrates the use of Borsuk-Ulam and Tucker's Theorems to construct a consensus-halving division. This paper actually provides a constructive version of a previous result by Alon [1].

The cake-cutting problem has received attention in Computer Science. The problems consist of turning cutting procedures into actual algorithms, designing algorithms which minimize the number of cuts [7, 8, 9, 12], and proving related lower bounds [17].

References

[1] N. Alon, Splitting Necklaces, Advances in Mathematics 63, 247-253 (1987).

[2] J. A. Bondy, U.S.R. Murthy, Graph Theory with Applications, North Holland, 1976.

[3] K. C. Border, Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge, UK: Cambridge University Press (1985).

[4] S. Brams, A.D. Taylor. Fair division: from cake cutting to dispute resolution. Cambridge University Press, 1996.

[5] S. Brams, A.D. Taylor, An Envy-Free Cake Division Protocol. Amer. Math. Monthly 102, 9-19, 1995.

[6] V. P. Crawford, A game of fair division, The Review of Economic Studies, Vol. 44(2), 235-247 (1977).

[7] J. Edmonds, K. Pruhs, Cake cutting really isnt a piece of cake. In: Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2006) (2006).

[8] J. Edmonds, K. Pruhs, Balanced Allocations of Cake. In: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2006), pp. 623634 (2006).

[9] J. Edmonds, K. Pruhs, J. Solanki, Confidently Cutting a Cake into Approximately Fair Pieces, in Algorithmic Aspects in Information and Management, Lecture Notes in Computer Science 5034 (2008).

[10] D. Gale, Dividing a Cake. Math. Intel. 15, 50, 1993.

[11] J. Robertson, W. Webb, Cake Cutting Algorithms: Be Fair If You Can. Wellesley, MA: A K Peters, 1998.

[12] J. Sgall, G. J. Woeginger, An Approximation Scheme For Cake Division With A Linear Number Of Cuts. Combinatorica 27(2): 205-211 (2007).

[13] F.W. Simmons, F.E. Su, Consensus-halving via theorems of Bursuk-Ulam and Tucker, Mathematical Social Sciences 45, 15-25 (2003).

[14] E. Sperner, Neuer Beweis f ur die Invarianz der Dimensionszahl und des Gebietes. Abh. Math. Sem. Hamburg VI (1928) 265272.

[15] H. Steinhaus, The problem of fair division. Econometrica 16, pp. 101-104 (1948).

[16] F. Su, Rental harmony: Sperner's lemma in fair division. Amer. Math. Monthly, 106(1999), pp. 930-942.

[17] G. J. Woeginger, J. Sgall, On the complexity of cake cutting. Discrete Optimization 4(2): 213-220 (2007)