Lecture 2: Examples

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We will present some examples of games with a few players and a few strategies. Each example will show some peculiarity of various solution concepts.

2.1 Two-player games in normal form

Games are a simplification of real world interactions, and can be used to analyze some features of such interactions. Let us start with some examples of *two-player* games in *normal form*, where the game is described in terms of two matrices, containing the *payoffs* of the two players.

Example 2.1 (Fair Division) *Alice and Bob have to split a cake. Alice (A) cuts the cake in two parts. Bob (B) chooses the piece he wants. A can either cut the cake into equal parts (strategy H) or cut it into unequal pieces (strategy U); B can decide to get either the larger piece (strategy L) or the smaller piece (strategy S). This set-up gives rise to the following bimatrix game. The first matrix describes the payoff of A, and the second one the payoff of B.*

	L	S
H	$HALF$	$HALF$
U	$SMALL$	BIG

	L	S
H	$HALF$	$HALF$
U	BIG	$SMALL$

The optimal strategy for A is to play H, and for B to play L, which leads to the jointly optimal outcome $(H, L) = (HALF, HALF)$*. Such combination of strategies is the unique Nash Equilibrium (NE) for this game.*

Example 2.2 (Prisoner's Dilemma) *Two suspects are arrested by the police. The police have insufficient evidence for a conviction, separate the prisoners, and visit each of them to offer the following deal. If one testifies against the other, and the other remains silent, the betrayer goes free and the accomplice receives a ten-year sentence. If both remain silent, both prisoners are sentenced to only one year in jail. If each betrays the other, each receives a five-year sentence. Each prisoner must choose either to betray the other (strategy C - confess) or to remain silent (strategy D - deny). Each one is assured that the other would not know about the betrayal before the end of the investigation.*

Here are the payoff matrices for this classical game. The entries of the matrices represent years in jail, and each player aims at minimizing jail time.

	C	D
C	5	0
D	10	1

	C	D
C	5	10
D	0	1

The unique NE is given by the pair of strategies (C, C)*, which gives a payoff of* 5 *to each player. Note that the "superior" outcome* $(1, 1)$ *is not a NE.*

The two examples above present some common features. They both have one NE, and, in both cases, the strategy leading to the NE can be chosen by each player independently of what the other player does (it is a *dominant strategy*).

However the two games differ in other - quite substantial - ways:

- The Fair Division game is a *constant sum game*, i.e., the sum of the two payoff matrices is a matrix with constant entries. Prisoner's dilemma is a *non-constant sum game*. Total jail time depends on what the players do, while the cake stays the same.
- In Prisoner's Dilemma both players have the same set of strategies, while in the Fair Division game the players' strategies are disjoint. Prisoner's Dilemma is a *symmetric game*, i.e. the payoff matrix of one player is the transpose of the payoff matrix of the other player.
- The NE in Prisoner's Dilemma is *inferior* to the outcome (D, D), where both players deny. However this is not an equilibrium point, as both players have an incentive to unilaterally deviate from it, and try to get by with a payoff of 0. The NE in the Fair Division game is not *dominated* by any other outcome: there is no outcome for which both players would be better off.

Example 2.3 (Chicken Game) *Consider two drivers headed for a single lane bridge from opposite directions. The first to stop aside yields the bridge to the other. If neither player yields, the result is a deadlock in the middle of the bridge, or a collision.*

This situation can be modeled by the following game, where each player has to choose between strategy Y (yield) and G (go).

	Y	G
Y	0	-1
G	1	-10

	Y	G
Y	0	1
G	-1	-10

The main principle governing this game is that each player prefers not to yield to the other, but the outcome where neither player yields is the worst possible for both players. This game has two NE, namely (Y, G) *and* (G, Y).

Example 2.4 (Battle of the Sexes) *Consider two partners, one would most of all like to go to the football game. The other one would like to go to a concert. But both of them would prefer to go to the same place rather than to different ones. This situation can be modeled by the following game, where each player has to choose between strategy F (football) and C (concert).*

	F	C
F	2	0
C	0	1

	F	C
F	1	0
C	0	2

The Battle of the Sexes is a coordination game*. This game has two NE, namely* (F, F) *and* (C, C).

Note that the Chicken game and the Battle of the Sexes both have two pure strategy NE. These two games are one the opposite of the other: while the Battle of the Sexes embodies the need for coordination, the Chicken game is a game of conflict, and is often called an *anti-coordination game*.

Example 2.5 (The Security Dilemma) *Consider two countries, which can choose between staying with the current defense system (strategy C) or move to a more expensive system (strategy D). Each of them would prefer to choose C if the other does so as well, but C would become very dangerous if the other decided for D.*

This situation can be modeled by the following game.

	C	D
C	9	0
D	8	7

	C	D
C	9	8
D	0	7

This game has two NE, namely (C, C) *and* (D, D)*.* (C, C) *is* Pareto-dominant*, while* (D, D) *is* risk dominant.

All the examples seen so far encompass games with at least one NE. We now show that there are games without any NE.

Example 2.6 (Matching Pennies) *Each of the two players has to either choose heads (H) or tails (T). The fist player wins if the outcome is either* (H, H) *or* (T, T)*, and the second player wins otherwise. This game can represent a situation where the two players at the same time call a number between* 1 *and* n*. If the sum of the numbers is even then the row player receives one dollar from the column player, and viceversa, if the sum is odd.*

	H	T
H	1	-1
T	-1	1

	H	T
H	-1	1
T	1	-1

This is a strictly competitive *game, where the interests of the players are totally opposed.*

It is easy to see that for each of the possible outcomes - (H, H)*,* (H, T)*,* (T, H)*, and* (T, T) *- one of the players has an incentive to unilaterally deviate. Thus there is no pure strategy NE for this game.*

As we will see below, the way out in this situation is to allow players to use randomization.

2.2 Mixed Strategies

The Matching Pennies game shows that there are games without a NE in the *pure strategies.* Nash has proved that a NE always exists in the mixed strategies [8, 9].

Example 2.7 (Matching Pennies in the Mixed Strategies) *Assume that the row player assigns probability* p *to strategy* H*, and* $1 - p$ *to* T*, while the column player assigns probability* q *to strategy* H*, and* $1 - q$ *to* T*.*

This combination of choices gives the following expected outcome:

$$pq(H,H) + (1-p)q(T,H) + p(1-q)(H,T) + (1-p)(1-q)(T,T).$$

Substituting the corresponding matrix entries in the above expression, we get the formulas for the expected payoffs, i.e., $(1-2p)(1-2q)$ *for the row player, and of course* $-(1-2p)(1-2q)$ *for the column player.*

$p = q = \frac{1}{2}$ *gives a NE for this game. Indeed, if* $q = \frac{1}{2}$*, then the row player does not have an incentive to deviate from* $p = \frac{1}{2}$*, because no other value of* p *is increasing her expected payoff. The same occurs for the other player. It is easy to check that this is the only NE for the game.*

Example 2.8 (Analysis of a simplified version of poker) *Consider the following very simplified version of poker. There is a deck of only two cards, A (the winning card), and K (the losing card), and two players (I and II). The deck is shuffled. Player I starts by picking a card from the deck. After looking at the card, I can either pass (strategy P) or raise (strategy R). If I passes, then I has to give II one dollar, and the game is over. If I raises, then it becomes II's turn, who can either pass and give a dollar to II or see (strategy S). If II chooses strategy S, then I has to show her card. If I has card A, then II has to give two dollars to I; if I has card K, then I has to give two dollars to II.*

We now represent this game in strategic form. I has four possible strategies, i.e., $P_A P_K$ *(always pass),* $P_A R_K$ *(pass on A and raise on K),* $R_A P_K$ *(raise on A and pass on K),* $R_A R_K$ *(always raise). II has two strategies, P, and S.*

We can now write up the matrices which describe this bimatrix game.

	P	S
$P_A P_K$	-1	-1
$P_A R_K$	0	-$\frac{3}{2}$
$R_A P_K$	0	$\frac{1}{2}$
$R_A R_K$	1	0

	P	S
$P_A P_K$	1	1
$P_A R_K$	0	$\frac{3}{2}$
$R_A P_K$	0	-$\frac{1}{2}$
$R_A R_K$	-1	0

To see how the entries have been filled, let us look at the combination of strategies $P_A R_K$ *and* P*. If I has A (which happens with probability* $\frac{1}{2}$*) the outcome is* $(-1, 1)$*, else the outcome is* $(1, -1)$*. The expected value of the payoff is thus* $(0, 0)$*. For the other combinations of strategies, we use similar arguments.*

It is easy to see that player I will never play the two strategies corresponding to the first two rows of the matrices, because they are dominated by other rows. Thus the game reduces to the form:

	P	S
$R_A P_K$	0	$\frac{1}{2}$
$R_A R_K$	1	0

	P	S
$R_A P_K$	0	-$\frac{1}{2}$
$R_A R_K$	-1	0

As expected, we immediately see that this game does not have pure strategy NE. Let us assume that player I plays the first row with probability p *(and thus that she is* bluffing *with probability* $1-p$*), while player II plays the first column with probability* q*. The expected gain for the first player is* $g_I = \frac{1}{2}p(1-q) + (1-p)q$*, and that of the second player is* $g_{II} = -\frac{1}{2}p(1-q) - (1-p)q$*.*

We now show that a NE occurs for $p = \frac{2}{3}$*, and* $q = \frac{1}{3}$*. When the second player chooses* $q = \frac{1}{3}$*, the expected payoff to I becomes* $\frac{1}{3}$ *(independently of* p*), and thus player I does not have an incentive to deviate* *from his current choice. Similarly, when the player I chooses* $p = \frac{2}{3}$*, the expected payoff to II becomes* $-\frac{1}{3}$ *(independently of* q*), and thus player II does not have an incentive to unilaterally deviate. Therefore, a NE occurs when I bluffs* $\frac{1}{3}$ *of the times and II passes* $\frac{1}{3}$ *of the times.*

2.3 Cooperative Games

In Lecture 1, we have seen an example of a cooperative game with an empty core. We now present a variation along the same theme (three persons having to share the profit of a job which can be done by two of them) where the core is not empty.

Example 2.9 (Asymmetric Coalitional Game) *Let us assume that the first player is more powerful than the other two, so that the function* v *can be redefined as* $v(N) = v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = v\{i\}) = 0$.

We then have:

$$\begin{cases} x_1 + x_2 \geq 1 \\ x_1 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

Adding up the first two inequalities, we get $2x_1 + x_2 + x_3 \geq 2$*, from which we have* $x_1 + (x_1 + x_2 + x_3) = x_1 + 1 \geq 2$.

This gives the solution $(1, 0, 0)$*, which is the unique imputation in the core.*

All other allocations give something *to player 2 and/or 3. For this reason, they cannot be in the core.*

The intuition is that, no matter how little player 2 gets, player 3 can start a bargaining game *with player 1, where both herself and player 1 are better off. For instance, if the allocation is* $(\frac{3}{4}, \frac{1}{4}, 0)$*, then player 3 may suggest* $(\frac{4}{5}, 0, \frac{1}{5})$*, and the process can continue.*

Coalition games where the goal of the coalitions is just to win are known as *simple games* or *weighted majority games* [14]. They are characterized by a payoff function $v(\cdot)$ which satisfies $v(S) = 1$ if the coalition S is winning and $v(S) = 0$ if it is losing.

Starting with the seminal paper by Shapley and Shubik [13], many quantities have been defined in cooperative GT to measure the power of coalitions when modeling decision processes, like elections and voting committees. These quantities are known as *power indices.*

Some of the bodies where decisions are taken by voting are designed to give different amounts of influence over the decisions to different participants.

Examples are

- joint stock companies, which give each shareholder a number of votes proportional to the quantity of ordinary stocks she owns;
- international organizations, where the countries' influences depend on their population;
- the International Monetary Fund, where the countries' influences depend on the amount of their contribution to the fund.

These systems are thus characterized by the weights assigned to the participants. Sometimes, the actual decision process occurs according to a distribution of power not intended by the designers, i.e., the distribution of power does not correspond to the distribution of votes (the weight).

Example 2.10 (The Council of the European Community) *In the Council of the European Economic Community there were originally (between 1958 and 1972) six members: France, West Germany, and Italy (each with four votes), Belgium and The Netherlands (each with two votes), and Luxembourg (with one vote). The quota (number of votes to reach a decision) was fixed to twelve. One could think that one vote for Luxembourg would be too much compared with the four votes of West Germany (given that the population of Luxembourg was just* 0.57% *of that of West Germany). But it is immediate to see that the actual influence of Luxembourg on any decision was zero: its vote could never make any difference.*

In general we represent weighted majority games as $[q; w_1, \ldots, w_n]$, where q is the quota, and the w_i's are the weights of the players.

Example 2.11 (Weight vs Power) *The game* $[5; 2, 3, 4]$ *is the same as the game* $[2; 1, 1, 1]$.

Example 2.12 *Consider the weighted majority game* $[51; 50, 49, 1]$.

The game can be described by the Boolean function $f(a, b, c) = a \wedge (b \vee c)$, *where the Boolean assignments to* a, b, *and* c *describe each possible coalition, and, for each coalition,* $f = 1$ *if and only if the coalition is winning.*

We have

a	b	c	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

2.4 Mechanism Design

We now present a typical problem of designing a mechanism that achieves a certain social goal, where we clearly see the role of mechanism design as the *engineering side* of Economic Theory.

Example 2.13 (The Energy Authority) *Consider a society of two individuals, Alice and Bob. An authority for energy has to choose the kind of energy to be used by Alice and Bob. The options are: gas, oil, nuclear energy, and coal. We assume that the society can be in two possible* states*:*

- *In state 1, the consumers put little emphasis on the future.*
- *In state 2, the consumers give much importance to the future.*

Alice is mostly concerned with her own comfort. In state 1, her preferences are gas > oil > coal > nuclear. In state 2, her preferences are nuclear > gas > coal > oil, because she imagines that technological progress will make nuclear, gas, and coal (in order) easier to use.

Bob is instead mainly interested in safety. In state 1, his preferences are nuclear > oil > coal > gas. In state 2, his preferences are oil > gas > coal > nuclear, because he is worried about nuclear waste.

Let us assume that the authority would like to make both consumers reasonably happy, for instance, by looking for a choice that guarantees that each individual gets at least his second best. Thus, in state 1, the best choice would be oil, and in state 2 gas. In the language of Implementation Theory[1]*, if f is the social choice function, we have that $f(state1) = oil$, and $f(state2) = gas$.*

Assume however that the state of the society is known only to Alice and Bob, and not to the authority. The simplest mechanism would be for the authority to ask each consumer, and choose oil if they both report state 1, gas if they report state 2, and toss a coin in case of a disagreement. However, in this mechanism Alice has an incentive to always report state 2, because she prefers gas to oil in both states. Similarly, we expect Bob to always report state 1, because he prefers oil to gas in both states.

Taken together, the behaviors of Alice and Bob imply that in each state the outcome would be a $50-50$ randomization between oil and gas. Therefore the outcome is optimal only with probability $\frac{1}{2}$.

We now show a less naive mechanism. The authority asks Alice and Bob to play a game, where Alice is the row player, Bob the column player, and they share the following payoff matrix.

	L	R
U	*oil*	*coal*
D	*nuclear*	*gas*

In state 1, Bob prefers to choose L, independently of what Alice does. Given that Bob will choose L, Alice will choose U, because she prefers oil to nuclear. Therefore in state 1 the pair of strategies (U, L) gives the only NE which leads to the outcome oil. Thus, in state 1, the NE coincides with the desired social choice.

Reasoning in the same way for state 2, we see that in this case the only NE is given by the pair (D, R), which again provides the desired outcome, i.e., gas.

It is remarkable that the mechanism of the simple game above achieves the desired outcome, in spite of the fact that the authority does not know the effective state, and both Alice and Bob are only motivated by their own preferences, and not by the socially desirable outcome.

Since the NE coincides with the optimal outcome in both states, we say that the mechanism implements the social choice rule of the authority in NE.

Bibliographic notes

The Fair Division Example is the starting point of the illustration of the main properties of *zero-sum* games and their connection to Linear Programming in [2].

For the Security Dilemma, see [1, 5]. The other examples of Section 2.1 can be found in most Game Theory textbooks, see for example Chapter 5, section 3 of [3], Chapter 1, section 2.4 of [4], and Chapter 2, section 2.3 of [10]. The analysis of the simplified version of poker is adapted from [12], pp. 37-41.

For cooperative games, the first example is taken from [10]. For the example about the European Council, see [6]. The Energy Authority case-study of Section 2.4 is part of the Nobel Prize Lecture by Maskin [7].

[1] Implementation Theory is a branch of Mechanism Design which has the goal of characterizing the problems for which a given social goal is implementable by a mechanism whose equilibria coincide with the desiderata.

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