In this lecture, we consider a network design problem, called the fair connection game, and analyze in this context the price of stability, i.e., the ratio between the best Nash equilibrium and the cost of the centrally designed optimum.

### 15.1 The Potential Function

Consider a strategic game with \( n \) players. If there is a function \( \Phi \) that associates with any strategy profile \( S = (S_1, \ldots, S_n) \) a real number such that, given two strategy profiles \( S = (S_1, \ldots, S_i, \ldots, S_n) \) and \( S' = (S_1', \ldots, S'_i, \ldots, S_n) \), the change \( \Phi(S) - \Phi(S') \) is equal to the change in utility (or cost) to player \( i \), we say that \( \Phi \) is the potential function of the game \( G \), and we call the game a potential game.

An important property of potential games is that they always have a NE in the pure strategies.

It turns out that any strategic game with homogeneous players where the payoff of each player depends on the number of players choosing each alternative is a potential game.

The potential function is a very useful tool for bounding the price of stability.

### 15.2 The Fair Connection Game

Consider the Fair Connection Game for \( n \) players. Let \( G = (V, E) \) be a directed graph, and let \( c_e \) be a nonnegative cost associated with each edge \( e \in E \). Player \( i \) owns a set \( T_i \) of nodes that she wants to connect, for \( i = 1, \ldots, n \). A strategy for player \( i \) is a set of edges \( S_i \subseteq E \) such that \( S_i \) connects all the nodes in \( T_i \).

We assume that the cost of the edges is shared equally by the players who use them. This cost-sharing scheme is called Shapley cost sharing mechanism, because it can be derived from the Shapley value of the game.

Consider a profile of strategies \( S = (S_1, S_2, \ldots, S_n) \). Then the cost to player \( i \) is given by \( C_i(S) = \sum_{e \in S_i} \frac{c_e}{n_e} \), where \( n_e \) denotes the number of players whose strategy contains edge \( e \).

The goal of each player is to connect her nodes with minimum cost.

The costs induced by a NE for this game can be very high, and the price of anarchy (the ratio between the cost of the worst NE and the cost of the optimal solution) turns out to be proportional to the number of players, as the following example shows.

**Example 15.1** Consider a directed graph with one source node and one sink node, and two directed edges connecting the source with the sink. Let \( 1 + \epsilon \) be the cost of one edge, and \( n \) be the cost associated with the other edge. Every player \( i, i = 1, 2, \ldots, n \), wants to connect the source and the sink. The optimal solution routes all the requests via the edge with cost \( 1 + \epsilon \). The solution where all the traffic goes through the edge with cost \( n \) is a NE. In fact each player has a cost of one. No player has an incentive to deviate and use the other edge, whose cost is \( 1 + \epsilon \).
Consider now the following example, which describes an instance of the Fair Connection game where the price of stability is high.

**Example 15.2** In the graph of Figure 15.1, player \( i \) wants to connect \( s_i \) to \( t_i, i = 1, \ldots, n \). The optimal solution routes all the requests through the edge with cost \( 1 + \epsilon \). Note that this is not a NE, since player \( n \) has an incentive to deviate and use the other path, which costs \( \frac{1}{n} \). If he so chooses, then player \( n - 1 \) has an incentive to deviate, and choose the edge with cost \( \frac{1}{n-1} \), and so on. This process leads to the unique NE, whose cost is \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \).

![Figure 15.1: An instance of the Fair Connection game with large price of stability.](image)

The result shown in Example 15.2 is the worst possible, as the following theorem shows.

**Theorem 15.3** The ratio between the cost of the best NE and the cost of the optimal solution (the price of stability) of the Fair Connection Game with \( n \) players is at most \( H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \).

**Proof:** The fair connection game falls into the class of potential games. Consider a potential function, defined as \( \Phi(S) = \sum_{e \in E} c_e H(n_e) \), where \( c_e \) is the cost of edge \( e \).

Consider the strategy profile \( S^* = (S_1^*, \ldots, S_n^*) \) defining the optimal solution. Let \( OPT = \sum_{e \in S^*} c_e \) be the cost of the optimal solution. \( \Phi(S^*) \leq \sum_{e \in S^*} (c_e \cdot H(n)) = H(n) \cdot OPT \). If we now start from \( S^* \) and follow a sequence of individually improving strategy changes, we end up at a NE \( S \) such that \( \Phi(S) \leq \Phi(S^*) \). Since \( \Phi(S) \geq \sum_{e \in S} c_e = Cost(S) \), we have that the cost at the NE \( S \) is at most \( H(n) \cdot OPT \). □

**Bibliographic notes**

Potential functions have been first used in Game Theory by Rosenthal in [4]. Rosenthal defined the class of congestion games and proved that they always have a pure strategy NE, by constructing a potential function. Monderer and Shapley introduced potential games, and proved several important properties about them [3]. In particular they showed that congestion games coincide with finite potential games.

The results on the fair connection problem have appeared in [2]. Related results had been previously shown in [1].
References


