Computational Aspects of Game Theory	**Bertinoro Spring School 2011**

Lecture 12: Scarf's Algorithm

Lecturer: Bruno Codenotti

In this lecture we will first present Scarf's algorithm for the approximation of a fixed point, and its application to the computation of market equilibria, and then illustrate the main ideas behind the more efficient path-following techniques.

12.1 A constructive proof of Sperner's lemma

Consider a restricted simplicial subdivision of the unit $(n-1)$-simplex S_n with no subdivision along the boundaries. Let $v^1, v^2, \ldots, v^n, v^{n+1}, \ldots, v^{n+k}$ be the vertices of the subdivision, where $v^1, v^2, \ldots, v^n$ are the unit vectors of the initial simplex S_n. We will associate a label $l(v) \in \{1, 2, \ldots, n\}$ with each vertex v.

Let $l(v^i) = i$, for $i = 1, 2, \ldots, n$. All the remaining vertices are in the interior of the simplex, and thus can be labeled arbitrarily from the set $\{1, 2, \ldots, n\}$.

We show now a constructive proof of the existence of a completely labeled simplex in the restricted subdivision above.

1. We start from the unique simplex of the subdivision whose vertices are $v^2, \ldots, v^n$, and an additional vertex v^{n+j}. We call this the initial simplex.

2. If $l(v^{n+j}) = 1$, then the algorithm ends since the current simplex is completely labeled. Otherwise $l(v^{n+j}) = k \neq 1$, and the label k appears twice in the current simplex.

3. We eliminate the vertex whose label coincides with the label of v^{n+j}. This will take us to another simplex, with a new vertex v^{n+k}. We now repeat step (2).

Note that at each stage of the algorithm we are at a simplex with the $n-1$ labels $\{2, \ldots, n\}$. termination only occurs when we are at a simplex with all the n labels $\{1, 2, \ldots, n\}$.

We now show that the algorithm never returns to a simplex it has already visited. This shows that the algorithm reaches a completely labeled simplex after a finite number of steps.

The proof is by contradiction. Consider the first simplex to be revisited. Let it be S_t. If it is not the initial simplex, then it can be reached in two ways, through either one of the adjacent simplices with $n-1$ distinct labels. Each of these adjacent simplices were already reached during the first visit, so that one of them is revisited before S_t, which cannot be the first simplex to be revisited. A similar argument holds if S_t is the initial simplex.

This constructive proof can be extended to prove Sperner's lemma. The idea is to embed a given unit $(n-1)$-simplex S_n (with an arbitrary simplicial subdivision) into a larger unit $(2n-1)$-simplex whose restricted subdivision extends the original general subdivision of S_n.

Therefore the computational scheme outlined above provides a constructive proof of Sperner's lemma.

12.2 Scarf's Algorithm

From the constructive proof of Sperner's lemma, we derive now an algorithm for the approximation of a fixed point of a function f mapping S_n into itself, using a labeling as in the proof of Brouwer's Theorem of lecture 4.

We associate with each vertex v^j an index i such that $v_i^j > 0$, and $f_i(v^j) \leq v_i^j$. With this labeling, a completely labeled simplex is such that at each of its vertices a different coordinate is not increased by f.

As we add vertices to the subdivisions of the simplex, the subdivisions become more refined, and these vertices may be selected so that the maximum diameter of the simplices tends to zero. Each subdivision contains a completely labeled simplex, and there exists a subsequence whose vertices converge to a single point x^*. This process is non-computational: we invoke the Bolzano-Weierstrass Theorem to argue the existence of a converging subsequence. From a quantitative point of view this process does not give a practical way to actually locate the fixed point.

Definition 12.1 (Weak Approximation) *Given a function f, we say that a point x is a weak approximation to a fixed point of f if $||x - f(x)||_\infty \leq \epsilon$. We say that x is an ϵ-fixed point.*

Note that an ϵ-fixed point does not need be close to an actual fixed point.

Definition 12.2 (Strong Approximation) *Given a function f, we say that a point x is a strong approximation to a fixed point of f if $||x - x^*||_\infty \leq \epsilon$, and $x^* = f(x^*)$.*

The definition of a strong approximation to a fixed point requires to determine a region of small diameter where the fixed point must necessarily lie. This approach is not computationally feasible for general functions, since it requires to anticipate the limit of a sequence from a finite amount of data. Scarf's algorithm rather provides a weak approximation to a fixed point.

Theorem 12.3 *Let G be a subdivision of S_n with mesh size at most δ. Let $f : S_n \to S_n$ be a continuous function such that $||x - z||_\infty \leq \delta$ implies $||f(x) - f(z)||_\infty \leq \epsilon$. Label each vertex x of G by $i = \min\{j : f_j(x) \leq x_j > 0\}$. If s is a completely labeled simplex of G, and $x^* \in s$, then $||f(x^*) - x^*||_\infty \leq n(\epsilon + \delta)$.*

Proof: Consider a completely labeled simplex s, and for each i, let y^i be the vertex in s with label i. For $x^* \in s$, we can write $f_i(x^*) - x_i^*$ as $(f_i(x^*) - f_i(y^i)) + (f_i(y^i) - y_i^i) + (y_i^i - x_i^*)$. The first of the three terms is upper bounded by ϵ, the second is nonpositive (by the labeling rule), and the third is upper bounded by δ. Therefore $f_i(x^*) - x_i^* \leq \epsilon + \delta$, and $f_i(x^*) - x_i^* \geq -n(\epsilon + \delta)$, and the theorem follows. ■

The bound of the Theorem can be improved.

12.3 Computation of Market Equilibria

We start by making some assumptions on the market excess demand function $z(\pi)$ (see lecture 5 for related discussions).

- z is well-defined, and continuous everywhere in the positive orthant, other than possibly at the origin.
- z is homogeneous of degree zero, i.e., $z(\alpha\pi) = z(\pi)$, for all $\alpha > 0$.

- z satisfies Walras' Law, i.e., for all π, $z(\pi) \cdot \pi = 0$.

In this context, an equilibrium price vector $\pi^* \geq 0$ satisfies $z_j(\pi^*) \leq 0$, for all j's, and $z_j(\pi^*) = 0$, when $\pi_j > 0$.

Consider a subdivision of the unit simplex S_n, with vertices $\pi^1, \pi^2, \ldots, \pi^n, \pi^{n+1}, \ldots, \pi^{n+k}$. We label each vertex π with an integer i such that $z_i(\pi) \leq 0$. To satisfy the conditions of a proper labeling, we must have that for all π there must exist i with $\pi_i > 0$ and $z_i(\pi) \leq 0$. This holds true because otherwise we would have that $\pi_i > 0$ implies $z_i(\pi) > 0$, which violates Walras' law.

Therefore there exists a completely labeled simplex that can be reached by the algorithm of the previous section.

Note that we can take a sequence of finer and finer subdivisions, select a converging subsequence of completely labeled simplices whose vertices tend to a price vector π^*. Since the excess demand function is continuous, we must have that $z_i(\pi^*) \leq 0$, for all i's.

12.4 More Efficient Algorithms

So far we have seen how to define a labeling from a function. We now see the converse, i.e., the definition of a function starting from a given labeling. This function will be the key concept behind the development of efficient *path-following* algorithms.

Consider the simplex $\tilde{S_n} = \{(x_1, \ldots, x_n) : \sum_i x_i \leq 1, x_i \geq 0\}$.

Definition 12.4 *Consider a simplicial decomposition of $\tilde{S_n}$ with vertices $v^1, v^2, \ldots, v^n, v^{n+1}, \ldots, v^{n+k}$, where $v^1, v^2, \ldots, v^n$ are the unit vectors of the initial simplex $\tilde{S_n}$. For each vertex v, let $l(v) = i \in \{1, 2, \ldots, n\}$ denote the label associated with it.*

We define the function $f : \tilde{S_n} \to \tilde{S_n}$ as:

1. *For each vertex v of the subdivision, set $f(v) = v^i$, where i is the label associated with v.*
2. *Having defined f on the vertices of the subdivision, we now extend it to the entire simplex $\tilde{S_n}$ by requiring it to be linear in each simplex of the subdivision.*

The two following properties of the function f are easily proven:

- f is piecewise linear in $\tilde{S_n}$.
- f is the identity map on the boundary of $\tilde{S_n}$.

Now assume that for any n-vector c *interior* to the simplex $\tilde{S_n}$ there exists x such that $f(x) = c$. The vertices of the simplex S_x containing x must bear distinct label. In fact, assume by contradiction that label i is missing, then the image under f of the vertices of S_x will be on the face of $\tilde{S_n}$ whose i-th coordinate is zero. Therefore $f(x)$ will lie on the boundary of $\tilde{S_n}$, which is a contradiction.

Conversely, if the vertices of S_x have all the labels, then, for every interior vector c, the system $f(x) = c$ has a solution contained in S_x.

Definition 12.5 *The polyhedron* $P \in R^{n+1}$ *is the product of the simplex* $\tilde{S_n}$ *with the closed interval* $[0,1]$*, i.e.,*

$$P = \{(x_1, x_2, \ldots, x_n, x_{n+1}) | x_i \geq 0, \sum_{i=1}^{n} x_i \leq 1, x_{n+1} \leq 1\}.$$

A subdivision of $\tilde{S_n}$ induces a subdivision of P into *pieces* P_i, each of which is obtained by taking the product of an elementary simplex with the closed interval $[0,1]$.

Consider now an n-vector d whose entries are all ones. Define from f a function $F : P \to R^n$, as

$$F(x_1, x_2, \ldots, x_n, x_{n+1}) = f(x_1, x_2, \ldots, x_n) - x_{n+1} d.$$

F is continuous and linear in each piece of the polyhedron P.

We want to show that, for an arbitrary vector c interior to $\tilde{S_n}$, $F^{-1}(c)$ will intersect the face of P for which $x_{n+1} = 0$.

First of all, note that $F^{-1}(c)$ can not intersect the face of P for which $x_{n+1} = 1$. In fact, this would imply $c = f(x_1, x_2, \ldots, x_n) - d$, which is impossible since c has all positive components, while $f(x_1, x_2, \ldots, x_n) \leq 1$, for all i's.

On the faces of P (other than those with $x_{n+1} = 0$ and $x_{n+1} = 1$) we know that $f(x) = x$. Therefore $x \in F^{-1}(c)$ must satisfy $x - x_{n+1} d = c$, which has the unique solution

$$(x_1^*, x_2^*, \ldots, x_n^*) = c + \frac{(1 - \sum_i c_i)}{n} d, \quad x_{n+1}^* = \frac{(1 - \sum_i c_i)}{n}.$$

We have shown that $F(x) = c$ has one solution on the boundary of P, except for the face where $x_{n+1} = 0$.

Before proceeding we need the following characterization of the solutions to $F(x) = c$.

Definition 12.6 (Regular value) *c is a degenerate value of F if there exists $x \in P$ lying on a face of dimension less than n of some piece P_i for which $F(x) = c$. A non-degenerate vector c is called a regular value of F.*

Theorem 12.7 (Eaves) *Let $F : P \to R^n$ be continuous and linear in each piece P_i, and let c be a regular value. Then the set of solutions of $F(x) = c$ is a finite disjoint union of paths and cycles, where each path intersects the boundary of P in precisely two points, and each loop does not have any intersection with the boundary.*

It is easy to show that there exists a vector c in the interior of $\tilde{S_n}$ which is a regular value of F.

We can then invoke Theorem 12.7 to produce a path starting from x^*. Since the path must end at some other boundary point of P, there must be a vector x on the face of P for which $x_{n+1} = 0$ and that satisfies $F(x) = c$.

It is easy to see that the path determined by the above construction coincides with the path followed by the simplicial algorithm described in Section 12.1.

There are several methods that can be used to follow such paths. The presentation of these methods is beyond the scope of these notes.

Bibliographic notes

A nice exposition of the constructive proof of Sperner's lemma can be found in [7]. The work by Scarf on the computation of equilibria has been presented in [3, 4, 5, 6]. A good introduction to fixed point computation in the economic context is [8].

The first results based on path following techniques are [1, 2].

The monograph by Todd contains a complete presentation of simplicial algorithms of various degree of sophistication for the computation of fixed points, and related economic applications [9].

References

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[9] M.J. Todd, The computation of fixed points and applications, Springer Verlag, New York (1976).